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3 (Sem–5/CBCS) MAT HC 1

2021

(Held in 2022)

MATHEMATICS

(Honours)

Paper : MAT–HC–5016

(Riemann Integration and Metric Spaces)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following as directed : 1×10=10

(a) Describe an open ball in the discrete metric space.

(b) Find the derived set of the sets $(0, 1]$ and $[0, 1]$.

(c) A subset B of a metric space (X, d) is open if and only if

(i) $B = \overline{B}$

(ii) $B = B^o$

(iii) $B \neq \overline{B}$

(iv) $B \neq B^o$

(Choose the correct one)

Contd.

(d) Which of the following is false ?

(i) $\phi^o = \phi,\ X^o = X$

(ii) $A \subseteq B \Rightarrow A^o \subseteq B^o$

(iii) $(A \cap B)^o = A^o \cap B^o$

(iv) $(A \cup B)^o = A^o \cup B^o$

where A, B are subsets of a metric space (X, d). *(Choose the false one)*

(e) The closure of the subset $F = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ of the real line $\mathbb{R}$ is

(i) ϕ

(ii) F

(iii) $F \cup \{0\}$

(iv) $F - \{0\}$

(Choose the correct one)

(f) In a metric space an arbitrary union of closed sets need not be closed. Justify it with an example.

(g) If A is a subset of a metric space (X, d), then which one is true ?

(i) $d(A) = d(\overline{A})$

(ii) $d(A) \neq d(\overline{A})$

(iii) $d(A) > d(\overline{A})$

(iv) $d(A) < d(\overline{A})$

(Choose the true one)

(h) When is an improper Riemann integral said to be convergent ?

(i) Evaluate $\int_0^\infty e^{-x} dx$ if it exists

(j) Show that $\Gamma(1) = 1$

2. Answer the following questions : 2×5=10

(a) Let F be a subset of a metric space (X, d). Prove that the set of limit points of F is a closed subset of (X, d).

(b) If F_1 and F_2 are two subsets of a metric space (X, d), then $\overline{F_1 \cap F_2} = \overline{F_1} \cap \overline{F_2}$. Justify whether it is false *or* true.

(c) Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. If for all subsets A of X, $f(\overline{A}) \subseteq \overline{f(A)}$, then show that f is continuous on X.

(d) Let $f : [a, b] \to \mathbb{R}$ be integrable. Show that $|f|$ is integrable.

(e) Show that the function $f : [a, b] \to \mathbb{R}$ defined by $f(x) = c$ for all $x \in [a, b]$ is integrable with its integral $c(b - a)$.

3. Answer ***any four*** parts : 5×4=20

(a) Define a complete metric space. Show that the metric space $X = \mathbb{R}^n$ with the metric given by

$$d_p(x, y) = \left(\sum |x_i - y_i|^p\right)^{\frac{1}{p}}, \quad p \geq 1$$

where $x = (x_1, x_2, \ldots, x_n)$ and

$y = (y_1, y_2, \ldots, y_n)$ are in $\mathbb{R}^n$, is a complete metric space. 1+4=5

(b) Let (X, d_X) and (Y, d_Y) be metric spaces. Prove that a mapping $f: X \to Y$ is continuous on X if and only if $f^{-1}(G)$ is open in X for all open subsets G of Y. 5

(c) Prove that if the metric space (X, d) is disconnected, then there exists a continuous mapping of (X, d) onto the discrete two-element space (X_0, d_0). 5

(d) Let $f:[a, b] \to \mathbb{R}$ be a continuous function. Prove that f is integrable. 5

(e) Discuss the convergence of the integral $\int_1^\infty \frac{1}{x^p} dx$ for various values of p. 5

(f) Show that for $a > -1$,

$$S_n = \frac{1^n + 2^n + \ldots\ldots + n^n}{n^{1+a}} \to \frac{1}{1+a}.$$ 5

4. Answer ***any four*** parts : 10×4=40

(a) (i) Let (X, d) be a metric space. Define $d: X \times X \to \mathbb{R}$ by

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)} \text{ for all}$$

$x, y \in X$.. Prove that d' is a metric on X.

Also show that d and d' are equivalent metrices on X.

4+2=6

(ii) Prove that a convergent sequence in a metric space is a Cauchy sequence. 4

(b) (i) Let (X, d) be a metric space and F be a subset of X. Prove that F is closed in X if and only if F^c is open. 5

(ii) If (Y, d_Y) is a subspace of a metric space (X, d), then show that a subset Z of Y is open in Y if and only if there exists an open set $G \subseteq X$ such that $Z = G \cap Y$.

5

(c) Prove that a metric space (X, d) is complete if and only if for every nested sequence $\{F_n\}_{n\geq 1}$ of non-empty closed subsets of X such that $d(F_n) \to 0$ as $n \to \infty$, the intersection $\bigcap_{n=1}^{\infty} F_n$ contains one and only one point. 10

(d) (i) Prove that in a metric space (X, d), each open ball is an open set. 4

(ii) Let (X, d_X) and (Y, d_Y) be metric spaces and $A \subseteq X$. Prove that a function $f : A \to Y$ is continuous at $a \in A$ if and only if whenever a sequence $\{x_n\}$ in A converges to a, the sequence $\{f(x_n)\}$ converges to $f(a)$. 6

(e) (i) Define uniformly continuous mapping in a metric space. Give an example to show that a continuous mapping need not be uniformly continuous. 1+4=5

Contd.

(ii) Prove that the image of a Cauchy sequence under a uniformly continuous mapping is itself a Cauchy sequence. 5

(f) Let $(\mathbb{R}, d)$ be the space of real numbers with the usual metric. Prove that a subset $I \subseteq \mathbb{R}$ is connected if and only if I is an interval. 10

(g) Let $f:[a,b] \to \mathbb{R}$ be a bounded function. Show that f is integrable if and only if it is Riemann integrable. 10

(h) (i) State and prove first fundamental theorem of calculus. Using it show that

$$\int_0^a f(x)dx = \frac{a^4}{4} \text{ for } f(x) = x^3.$$

1+3+2=6

(ii) Let f be continuous on $[a,b]$. Prove that there exists $c \in [a,b]$ such that $\frac{1}{b-a}\int_a^b f(x)dx = f(c)$.

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3 (Sem–5 /CBCS) MAT HC 2

2021

(Held in 2022)

MATHEMATICS

(Honours)

Paper : MAT–HC–5026

(Linear Algebra)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following as directed : 1×10=10

(i) Is $\mathbb{R}^2(\mathbb{R})$ is a subspace of $\mathbb{R}^3(\mathbb{R})$?

(ii) Let A be a 5×4 matrix. If null space of A is a subspace of $\mathbb{R}^k$ then what is k?

(iii) Let S be a subset of a vector space $V(F)$ and S contains zero vector of V. Then S is

(A) linearly independent

(B) linearly dependent

(C) Both linearly independent and linearly dependent

(D) None of the above

(Choose the correct option)

(iv) Write the standard basis of the vector space of polynomial in x with real co-efficient of degree ≤ 3.

(v) "The eigenvalues of a triangular matrix are the entries on its main diagonal." *(State True **or** False)*

(vi) Define inner product on $\mathbb{R}^n$.

(vii) Which vector is orthogonal to every vector in $\mathbb{R}^n$?

(viii) How do you explain $dim\ W = 1$ geometrically where W is a subspace of the vector space $\mathbb{R}^3(\mathbb{R})$?

(ix) Let A be the 4×4 real matrix,

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & -2 & 0 \\ 1 & 1 & -2 & 1 \end{bmatrix}$$

Then the characteristic polynomial for A is

(A) $x^2(x-1)^2$

(B) $(x-1)^2(x+1)^2$

(C) $x^2(x+1)^2$

(D) None of the above

(Choose the correct option)

(x) What do you mean by the length of a vector in $\mathbb{R}^n$?

2. Answer the following questions : $2 \times 5 = 10$

(i) Let V be the vector space of all functions from the real field $\mathbb{R}$ to $\mathbb{R}$. Show that $W = \{f : f(7) = 2 + f(1)\}$ is not a subspace of V.

(ii) Show that every subset of an independent set is independent.

(iii) Let $A=\begin{bmatrix}1 & 6\\ 5 & 2\end{bmatrix}$ and $v=\begin{bmatrix}3\\ -2\end{bmatrix}$. Is v a eigenvector of A ?

(iv) Let T be the linear operator on $\mathbb{R}^3$ defined by $T(a,b,c)=(a+b,\, b+c,\, 0)$. Show that the $xy\text{-plane}=\{(x,y,0): x,y\in\mathbb{R}\}$ is T-invariant subspace of $\mathbb{R}^3$.

(v) Let $y=\begin{bmatrix}7\\ 6\end{bmatrix}$ and $u=\begin{bmatrix}4\\ 2\end{bmatrix}$. Find the orthogonal projection of y onto u.

3. Answer ***any four*** questions : 5×4=20

(i) Prove that the non-zero vectors $v_1, v_2, \ldots, v_n$ are linearly dependent if and only if one of them is a linear combination of the preceding vectors.

(ii) Let $v_1, v_2, \ldots, v_n$ be non-zero eigenvectors of an operator $T: V \to V$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Prove that $v_1, v_2, \ldots, v_n$ are linearly independent.

(iii) Let A and B be two similar matrices of order $n \times n$. Prove that A and B have same characteristic polynomial and hence the same eigenvalues.

(iv) Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

(v) Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for A^2, given that $A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

(vi) Define orthogonal set. If $S = \{u_1, u_2, \ldots, u_p\}$ is an orthogonal set of non-zero vectors in $\mathbb{R}^n$, then prove that S is linearly independent and hence is a basis for the subspace spanned by S.

Contd.

4. *(i)* If a vector space V has a basis $B = \{v_1, v_2, ..., v_n\}$, then prove that any set in V containing more than n vectors must be linearly dependent. Also show that every basis of V must consist of exactly n vectors. 5+5=10

OR

Let U and V be vector spaces over the same field. Let $\{u_1, u_2, ..., u_n\}$ be a basis of U and let $v_1, v_2, ..., v_n$ be any arbitrary vectors in V. Prove that there exists a unique linear mapping $f : U \to V$ such that

$f(u_1) = v_1, f(u_2) = v_2, ..., f(u_n) = v_n$ 10

(ii) Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$. 10

OR

State Cayley-Hamilton theorem for matrices. Use it to express $2A^5 - 3A^4 - A^2 - 4I$ as a linear polynomial in A, when $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$. 10

(iii) Let T be the linear operator on $\mathbb{R}^3$, defined by

$$T(x,y,z) = (2y+z,\ x-4y,\ 3x)$$

(a) Find the matrix of T in the basis $\{e_1 = (1,1,1),\ e_2 = (1,1,0),\ e_3 = (1,0,0)\}$

(b) Verify that $[T]_e [v]_e = [T(v)]_e$ for any vector $v \in \mathbb{R}^3$. 4+6=10

OR

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigen-vectors. 10

(iv) Define orthonormal set and orthonormal basis in $\mathbb{R}^n$. Show that $\{u_1, u_2, u_3\}$ is an orthonormal basis of $\mathbb{R}^3$, where

$$u_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix},\quad u_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix},\quad u_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$$

1+1+8=10

OR

Define inner product space. Show that the following is an inner product in $\mathbb{R}^2$:

$$\langle u, v \rangle = x_1 y_1 - x_1 y_2 - x_2 y_1 + 3x_2 y_2$$

where $u = (x_1, x_2)$, $v = (y_1, y_2)$.

Also show that for all u, v in $\mathbb{R}^2$

$\|u + v\| \leq \|u\| + \|v\|$ 2+5+3=10