Total number of printed pages–7

3 (Sem–1/CBCS) MAT HC 1

2021

(Held in 2022)

MATHEMATICS

(Honours)

Paper : MAT–HC–1016

(Calculus)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : $1 \times 7 = 7$

   (a) Write down the nth derivative of $y = \log x$.

   (b) The point $P(c, f(c))$ on the graph of $f(x)$ is such that $f''(c) = 0$. Does it necessarily imply that P is an inflection point on the graph ?

Contd.

(c) Write down the value of $\underset{x \to +\infty}{lt} x \sin\frac{1}{x}$.

(d) Find the domain of the vector function

$$\vec{F}(t) = (1-t)\hat{i} + \sqrt{-t}\,\hat{j} + \frac{1}{t-2}\hat{k}$$

(e) Write one basic difference between the disk/washer and shell method for computing volume of revolution.

(f) What is the direction of velocity of a moving object on its trajectory.

(g) The velocity of a particle moving in space is $\vec{V}(t) = e^t\hat{i} + t^2\hat{j}$. Find the direction of motion at time $t = 2$.

2. Answer the following questions : 2×4=8

(a) Applying L.Hopital's rule, evaluate

$$\underset{x \to \pi/4}{lt} (1 - \tan x).\sec 2x$$

(b) Write down the parametric equation of a line that contains the point $(3,1,4)$ and is parallel to the vector $\vec{v} = -\hat{i} + \hat{j} - 2\hat{k}$.

(c) Find the area of the surface generated by revolving the portion of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the x-axis.

(d) Explain briefly why the acceleration of an object moving with constant speed is always orthogonal to the direction of motion.

3. Answer ***any three*** of the following questions : 5×3=15

(a) If $y = \cos\left(m \sin^{-1} x\right)$, show that

$$\left(1-x^2\right)y_{n+2} - (2n+1)x\, y_{n+1} + \left(m^2 - n^2\right)y_n = 0.$$

Hence find $y_n(0)$. 3+2=5

(b) Sketch the graph of a function f with all the following properties : 5

(i) the graph has $y=1$ and $x=3$ as asymptotes

(ii) f is increasing for $x<3$ and $3<x<5$ and decreasing elsewhere

(iii) the graph is concave up for $x<3$ and concave down for $3<x<7$

(iv) $f(0)=4=f(5)$ and $f(7)=2$

(c) Sketch the graph of $y = \frac{3x-5}{x-2}$ identifying the locations of intercepts, concavity and inflection points (if any) and asymptotes. 5

(d) Obtain the reduction formula for $\int tan^n x\, dx$.

Hence evaluate $\int_0^{\pi/4} tan^5 x\, dx$ 3+2=5

(e) The position vector of a moving object at any time t is given by $\vec{R}(t) = t\hat{i} + e^t \hat{j}$. Find the tangential and normal components of the object's acceleration. 5

4. Answer **any three** of the following questions : 10×3=30

(a) A firm determines that x units of its product can be sold daily at rupees p per unit where $x = 1000 - p$. The cost of producing x units per day is $C(x) = 3000 + 20x$. Then —

(i) Find the revenue function $R(x)$. 2

(ii) Find the profit function $p(x)$. 2

(iii) Assuming that production capacity is atmost 500 units per day, determine how many units the company must produce and sell each day to maximize profit. 3

(iv) Find the maximum profit. 2

(v) What price per unit must be charged to obtain maximum profit ? 1

(b) When is an object said to move in central force field ? Derive Kepler's 2nd law of motion, assuming that planetary motion occurs in central force field. 2+8=10

(c) (i) Find the length of the arc of the astroid $x^{2/3}+y^{2/3}=1$ lying in the positive quadrant. 3

(ii) Using cylindrical shell method, find the volume of the solid formed by revolving the region bounded by the parabola $y=1-x^2$, the y-axis, and the positive x-axis, about y-axis. 4

(iii) Find the surface area generated when the polar curve

$$r = 5,\ 0 \le \theta \le \pi/3$$

is revolved about x-axis. 3

(d) (i) Find the volume generated by disk/washer method, when the region bounded by $y = x$, $y = 2x$ and $y = 1$ is revolved about the x-axis 5

(ii) A particle moves along the polar path (r, θ) where

$$r(t) = 3 + 2\sin t,\ \theta(t) = t^3.$$

Find the velocity $\bar{v}(t)$ and acceleration $\vec{A}(t)$ in terms $\hat{u}_r$ and $\hat{u}_\theta$. 5

(e) (i) Evaluate $\lim\limits_{x \to 0} (1 + \sin x)^{1/x}$. 3

(ii) Examine the existence of vertical tangent and cusp of the graph of $y = (x - 4)^{2/3}$. 3

(iii) A projectile is fired from ground level at an angle of 30° with muzzle speed 110 ft/sec. Find the time of flight and the range. 4

(f) (i) Obtain the reduction formula for

$$\int \cos^n x \, dx.$$

Hence evaluate $\int \cos^5 x \, dx$.

3+2=5

(ii) Find the unit tangent vector $\vec{T}(t)$ and principal unit normal vector $\vec{N}(t)$ at each point on the graph of vector function

$$\vec{R}(t) = (3\sin t, 4t, 3\cos t)$$ 5

---

3 (Sem-1/CBCS) MAT HC 2

2021

(Held in 2022)

MATHEMATICS

(Honours)

Paper : MAT–HC–1026

(Algebra)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following as directed : $1 \times 10 = 10$

(a) Find the polar representation of $z = 2i$.

(b) If $x = 0$ and $y > 0$, then what is the value of t^* ?

(c) Write the negation of the statement 'For any integer n, $n^2 > n$' in plain English then formulate the negation using set of context and quantifier.

(d) Disapprove the statement using counter example :

"For any $x, y \in \mathbb{R}$, $x^2 = y^2$ implies $x = y$."

(e) Suppose f is a constant function from X to Y. The inverse image of a subset of Y cannot be

(i) an empty set

(ii) the whole set X

(iii) a non-empty proper subset of X

(Choose the correct option)

(f) Let $X = Y = \mathbb{R}$. Let $A \subseteq X$, $B \subseteq Y$. Draw the picture for $A \times B$ where $A = [-1, 1]$ and $B = [2, 3]$.

(g) Suppose a system of linear equations in echelon form has a 3×5 augmented matrix whose fifth column is a pivot column.

Is the system consistent? Justify.

(h) If a set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ in $\mathbb{R}^n$ contains the $\vec{O}$ vector, is the set linearly independent? Justify.

(i) If $A=\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$ $\vec{x}=\begin{bmatrix} 5 \\ 3 \end{bmatrix}$, compute $(A\vec{x})^T$.

(j) What is the determinant of an $n \times n$ elementary matrix E that has been scaled by 7.

2. Answer the following questions : $2\times5=10$

(a) If $z=-2\sqrt{3}-2i$, find the polar radius and polar argument of z.

(b) Is the function $g:\mathbb{R}\to\mathbb{R}$ given by $g(x)=|x-2|$ one-one and onto? Explain.

(c) Let universal set be $\mathbb{R}$ and index set be $\mathbb{N}$. For a natural number n, $J_n=\left(0,\frac{1}{n}\right)$.

Identify with justification $\bigcap_{n\in N} J_n$.

(d) Show that T is a linear transformation by finding a matrix that implements the mapping

$$T(x_1, x_2, x_3, x_4) = (0, x_1 + x_2, x_2 + x_3, x_3 + x_4)$$

(e) A is a 2×4 matrix with two pivot positions. Answer the following with justification :

(i) Does $A\vec{x} = \vec{0}$ have a non-trivial solution ?

(ii) Does $A\vec{x} = \vec{b}$ have at least one solution for every $\vec{b}$?

3. Answer ***any four*** questions from the following : $5 \times 4 = 20$

(a) Find the polar representation of the complex number
$z = 1 - \cos a + i \sin a \quad a \in [0, 2\pi)$. 5

(b) Let A and B be subsets of an universal set U. Prove —

(i) $(A \cap B)^C = A^C \cup B^C$

(ii) $(A \cup B)^C = A^C \cap B^C$ 5

(c) Define bijection.

Let $f:\mathbb{N}\to\mathbb{N}$ be $f(m)=m-1$, if m is even $f(m)=m+1$, if m is odd. Show f is a bijection and $f^{-1}=f$. 1+4=5

(d) For vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p \in \mathbb{R}^n$ define span $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ construct a 3×3 matrix A with non-zero elements and a vector $\vec{b}$ on $\mathbb{R}^3$ such that $\vec{b}$ is not in the set spanned by the columns of A. 2+3=5

(e) Alka-Seltzer contains sodium bicarbonate ($NaHCO_3$) and citric acid ($H_3C_6H_5O_7$). When a tablet is dissolved in water the following reaction produces sodium citrate, water and carbon dioxide :

$$NaHCO_3 + H_3C_6H_5O_7 \to Na_3C_6H_5O_7 + H_2O + CO_2$$

Balance the chemical equation using vector equation approach. 5

(f) Prove that an $n\times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}. 5

4. Answer ***any four*** from the following : $10\times4=40$

(a) (i) Find the cube roots of the number $z = 1 + i$ and represent them in the complex plane. 5

(ii) Find the number of ordered pairs (a, b) of real numbers such that $(a+ib)^{2002} = a - ib$. 2

(iii) If x, y, z be real numbers such that $\sin x + \sin y + \sin z = 0$ and $\cos x + \cos y + \cos z = 0$, prove that $\sin 2x + \sin 2y + \sin 2z = 0$ and $\cos 2x + \cos 2y + \cos 2z = 0$. 3

(b) (i) Solve the equation

$$z^7 - 2iz^4 - iz^3 - 2 = 0.$$ 5

(ii) Find the inverse of the matrix if it exists by performing suitable row operations on the augmented matrix $[A : I]$

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$ 5

(c) (i) If $f : X \to Y$ be a map and $B \subseteq Y$, then prove $f^{-1}\left(B^C\right) = \left(f^{-1}(B)\right)^C$. 4

(ii) $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$, where $n \in \mathbb{N}$. Find $\bigcup_{n \in N} A_n$ and $\bigcap_{n \in N} A_n$. 2

(iii) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$.

Find $f^{-1}(1), f^{-1}(-1), f^{-1}([0, 1])$. 4

(d) (i) State the induction principle and use it to show that for any positive integer $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$. 4

(ii) Write as an implication 'square of an even number is divisible by 4'. Then use direct proof to prove it. 3

(iii) Give proof using contrapositive 'For an integer x if x^2-6x+5 is even, then x is odd'. 3

(e) (i) Use the invertible matrix theorem to decide if A is invertible

$$A=\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 2 \\ -5 & -1 & 9 \end{bmatrix}$$ 2

(ii) Compute *det* A where

$$A=\begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$$ 4

(iii) What do you mean by equivalence class for an equivalence relation ? For the relation $a \equiv b \, mod(5)$ on z, find all the distinct equivalence classes of z. 1+3=4

(f) (i) Solve the system of equations

$$x_1 - 3x_3 = 8$$
$$2x_1 + 2x_2 + 9x_3 = 7$$
$$x_2 + 5x_3 = -2$$

4

(ii) Choose h and k such that the system has 4

(a) no solution

(b) a unique solution

(c) many solutions

$$x_1 + hx_2 = 2$$
$$4x_1 + 8x_2 = k$$

(iii) Write the general solution of $10x_1 - 3x_2 - 2x_3 = 7$ in parametric vector form. 2

(g) (i) Prove that the indexed set $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\vec{v}_1 \neq \vec{0}$, then some $\vec{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_{j-1}$. 5

(ii) Use Cramer's rule to compute the solutions of the system 3

$$-5x_1 + 3x_2 = 9$$
$$3x_1 - x_2 = -5$$

(iii) Suppose $T : \mathbb{R}^5 \to \mathbb{R}^2$ and $T(\vec{x}) = A\vec{x}$ for some matrix A and each $\vec{x}$ in $\mathbb{R}^5$.

How many rows and columns does A have ? Justify. 2

(h) (i) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle ϕ with the counter-clockwise direction taken as positive. Find the standard matrix for this transformation. 3

(ii) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

Prove that T is one-to-one if and only if the equation $T(\vec{x}) = \vec{0}$ has only the trivial solution. 4

(iii) Find the area of the parallelogram whose vertices are $(0, -2)$, $(6, -1)$, $(-3, 1)$ and $(3, 2)$. 3
