3 (Sem–3 /CBCS) MAT HC 1

2021

(Held in 2022)

MATHEMATICS

(Honours)

Paper : MAT-HC-3016

(Theory of Real Functions)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following as directed : $1\times10=10$

(a) Find $\lim\limits_{x\to 2}\dfrac{x^3-4}{x^2+1}$

(b) Is the function $f(x)=x\sin\left(\dfrac{1}{x}\right)$ continuous at $x=0$?

(c) Write the cluster points of $A=(0,1)$.

(d) If a function $f:(a,\infty)\to\mathbb{R}$ is such that $\lim\limits_{x\to\infty} xf(x)=L$, where $L\in\mathbb{R}$, then $\lim\limits_{x\to\infty} f(x)=?$

(e) Write the points of continuity of the function $f(x)=\cos\sqrt{1+x^2}$, $x\in\mathbb{R}$.

(f) "Every polynomial of odd degree with real coefficients has at least one real roof." Is this statement true **or** false ?

(g) The derivative of an even function is ________ function. *(Fill in the blank)*

(h) Between *any two* roots of the function $f(x)=\sin x$, there is at least ________ root of the function $f(x)=\cos x$.

(Fill in the blank)

(i) If $f(x)=|x^3|$ for $x\in\mathbb{R}$, then find $f'(x)$ for $x\in\mathbb{R}$.

(j) Write the number of solutions of the equation $ln(x)=x-2$.

2. Answer the following questions : $2\times5=10$

(a) Show that $\lim\limits_{x\to 0}(x+sgn(x))$ does not exist.

(b) Let f be defined for all $x\in\mathbb{R}$, $x\neq 3$ by $f(x)=\dfrac{x^2+x-12}{x-3}$. Can f be defined at $x=3$ in such a way that f is continuous at this point ?

(c) Show that $f(x)=x^2$ is uniformly continuous on $[0, a]$, where $a>0$.

(d) Give an example with justification that a function is 'continuous at every point but whose derivative does not exist everywhere'.

(e) Suppose $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=x^2\sin\dfrac{1}{x^2}$, for $x\neq 0$ and $f(0)=0$. Is f' bounded on $[-1,1]$?

3. Answer ***any four*** parts : 5×4=20

(a) If $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ has a limit at $c \in \mathbb{R}$, then prove that f is bounded on some neighbourhood of c.

(b) Let $f(x) = |2x|^{-\frac{1}{2}}$ for $x \neq 0$. Show that

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = +\infty.$$

(c) Show that the function $f(x) = |x|$ is continuous at every point $c \in \mathbb{R}$.

(d) Give an example to show that the product of two uniformly continuous function is not uniformly continuous on $\mathbb{R}$.

(e) Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. If f' is positive on $[a, b]$, then prove that f is strictly increasing on $[a, b]$.

(f) Evaluate —

$$\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$$

4. Answer ***any four*** parts : 10×4=40

(a) Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. Prove that the following are equivalent.

(i) $\lim_{x \to c} f(x) = l$

(ii) For every sequence (x_n) in A that converges to c such that $x_n \neq c$ for all $x \in \mathbb{N}$, the sequence $(f(x_n))$ converges to l. 10

(b) (i) Give examples of functions f and g such that f and g do not have limits at a point c but such that both $f+g$ and fg have limits at c. 6

(ii) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let c be a cluster point of A. If $\lim_{x \to C} f(x)$ exists and if $|f|$ denotes the function defined for $x \in A$ by $|f|(x) = |fx|$, Proof that

$$\lim_{x \to c} |f|(x) = \left| \lim_{x \to c} f(x) \right|$$ 4

(c) Prove that the rational functions and the sine functions are continuous on $\mathbb{R}$. 10

(d) (i) Let I be an interval and let $f: I \rightarrow \mathbb{R}$ be continuous on I. Prove that the set $f(I)$ is an interval. 5

(ii) Show that the function $f(x) = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$. 5

(e) State and prove maximum-minimum theorem. 2+8=10

(f) (i) If $f: I \rightarrow \mathbb{R}$ has derivative at $c \in I$, then prove that f is continuous at c. Is the converse true ? Justify. 6

(ii) If r is a rational number, let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Determine those values of r for which $f'(0)$ exists. 4

(g) State and prove Mean value theorem. Give the geometrical interpretation of the theorem. (2+5)+3=10

(h) State and prove Taylor's theorem.

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3 (Sem–3/CBCS) MAT HC 2

2021

(Held in 2022)

MATHEMATICS

(Honours)

Paper : MAT–HC–3026

(Group Theory–I)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×10=10

   (a) Give the condition on n under which the set $\{1, 2, 3, ..., n-1\}$, $n > 1$ is a group under multiplication modulo n.

   (b) Define a binary operation on the set $\mathbb{R}^n = \{(a_1, a_2, ..., a_n) : a_1, a_2, ..., a_n \in \mathbb{R}\}$ for which it is a group.

Contd.

(c) What is the centre of the dihedral group of order $2n$?

(d) Write the generators of the cyclic group $\mathbb{Z}$ (the group of integers) under ordinary addition.

(e) Show by an example that the decomposition of a permutation into a product of 2-cycles is not unique.

(f) Find the cycles of the permutation :

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 1 & 2 \end{pmatrix}$$

(g) Find the order of the permutation :

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 1 & 3 \end{pmatrix}$$

(h) Let G be the multiplicative group of all non-singular $n \times n$ matrices over $\mathbb{R}$ and let $\mathbb{R}^*$ be the multiplicative group of all non-zero real numbers. Define a homomorphism from G to $\mathbb{R}^*$.

(i) What do you mean by an isomorphism between two groups ?

(j) State the second isomorphism theorem.

2. Answer the following questions : $2\times5=10$

(a) Let G be a group and $a \in G$. Show that $\langle a \rangle$ is a subgroup of G.

(b) If G is a finite group, then order of any element of G divides the order of G. Justify whether this statement is true or false.

(c) Show that a group of prime order cannot have any non-trivial subgroup. Is it true for a group of finite composite order ?

(d) Consider the mapping ϕ from the group of real numbers under addition to itself given by $\phi(x) = [x]$, the greatest integer less than or equal to x. Examine whether ϕ is a homomorphism.

(e) Let ϕ be an isomorphism from a group G onto a group H. Prove that ϕ^{-1} is also an isomorphism from H onto G.

3. Answer the following questions : 5×4=20

(a) Show that a finite group of even order has *at least one* element of order 2.

Or

Let N be a normal subgroup of a group G. Show that G/N is abelian if and only if for all $x, y \in G$, $xyx^{-1}y^{-1} \in N$.

(b) Show that if a cyclic subgroup K of a group G is normal in G, then every subgroup of K is normal in G.

Or

Show that converse of Lagrange's theorem holds in case of finite cyclic groups.

(c) Consider the group $G = \{1, -1\}$ under multiplication. Define $f : \mathbb{Z} \to G$ by

$$f(x) = 1, \text{ if } n \text{ is even}$$
$$= -1, \text{ if } n \text{ is odd}$$

Show that f is a homomorphism from $\mathbb{Z}$ to G.

(d) Let $f : G \to G'$ be a homomorphism. Let $a \in G$ be such that $o(a) = n$ and $o(f(a)) = m$. Prove that $o(f(a))/o(a)$, and if f is one-one, then $m = n$.

4. Answer the following questions : 10×4=40

(a) Let G be a group and $x, y \in G$ be such that $xy^2 = y^3x$ and $yx^2 = x^3y$. Then show that $x = y = e$, where e is the identity element of G. 10

Or

3. Give an example to show that the product of two subgroups of a group is not a subgroup in general. Also show that if H and K are two subgroups of a group G, then HK is a subgroup of G if and only if $HK = KH$. 2+8=10

(b) Prove that the order of a cyclic group is equal to the order of its generator. 10

Or

Let H be a non-empty subset of a group G. Define $H^{-1} = \{h^{-1} \in G : h \in H\}$. Show that

(i) if H is a subgroup of G, then $HH = H$, $H = H^{-1}$ and $HH^{-1} = H$;

(ii) if H and K are subgroups of G, then $(HK)^{-1} = K^{-1}H^{-1}$. 5+5=10

(c) Let G be a group and $Z(G)$ be the centre of G. If $G/Z(G)$ is cyclic, then show that G is abelian. 10

Or

State and prove Lagrange's theorem. 10

(d) Let H and K be two normal subgroups of a group G such that $H \subseteq K$. Show that $G/K \cong \dfrac{G/H}{K/H}$. 10

Or

Prove Cayley's theorem. 10

Total number of printed pages–8

3 (Sem–3/CBCS) MAT HC 3

2021

(Held in 2022)

MATHEMATICS

(Honours)

Paper : MAT–HC–3036

(Analytical Geometry)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×10=10

(i) What is the nature of the conic represented by

$4x^2 - 4xy + y^2 - 12x + 6y + 9 = 0$?

(ii) Define skew lines.

Contd.

(iii) Under what condition

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

may represents a pair of parallel straight lines ?

(iv) If the axes are rectangular, find the direction cosines of the normal to the plane $x + 2y - 2z = 9$.

(v) Write down the conditions under which the general equation of second degree

$ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$

represents a sphere.

(vi) If $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is a generator of the cone represented by the homogeneous equation $f(x, y, z)$, then what is the value of $f(l, m, n)$?

(vii) What is meant by diametral plane of a conicoid ?

(viii) Find the equation of the line $\frac{x}{a} + \frac{y}{b} = 2$, when the origin is transferred to the point (a, b).

(ix) Find the point on the conic $\frac{8}{r} = 3 - \sqrt{2}\cos\theta$ whose radius vector is 4.

(x) What is the polar equation of a circle when the pole is at the centre ?

2. Answer the following questions : 2×5=10

(a) Write down the equation to the cone whose vertex is the origin and which passes through the curve of intersection of the plane $lx + my + nz = p$ and the surface $ax^2 + by^2 + cz^2 = 1$.

(b) Transform the equation $x^2 - y^2 = a^2$ by taking the perpendicular lines $y - x = 0$ and $y + x = 0$ as coordinate axes.

(c) If $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are the extremities of any focal chord of the parabola $y^2 = 4ax$, then prove that $t_1t_2 = -1$.

(d) Find the centre and foci of the hyperbola $x^2 - y^2 = a^2$.

(e) Find where the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4}$ meets the plane $x + y + z = 3$.

3. Answer **any four**: 5×4=20

(a) If by transformation from one set of rectangular axes to another with the same origin the expression $ax + by$ changes to $a'x' + b'y'$, prove that $a^2 + b^2 = a'^2 + b'^2$.

(b) Prove that the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a pair of parallel straight lines, if $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$.

(c) Find the condition that line

$$\frac{l}{r} = A\cos\theta + B\sin\theta$$

may touch the conic $\frac{l}{r} = 1 - e\cos\theta$.

(d) Find the equation to the plane which cuts $x^2 + 4y^2 - 5z^2 = 1$ in a conic whose centre is the point (2,3,4).

(e) Show that the equation to the cone whose vertex is origin and base is $z = k, f(x, y) = 0$ is $f\left(\frac{kx}{z}, \frac{ky}{z}\right) = 0$.

Contd.

(f) A variable plane is at a constant distance p from the origin and meets the axes, which are rectangular in A, B, C. Through A, B, C planes are drawn parallel to the coordinate planes, show that locus of their point of intersection is given by $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

4. Answer the following questions : 10×4=40

(a) Find the point of intersection of the lines represented by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

(b) Show that the equation
$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$$
represents a parabola and it can be reduced to the standard form $Y^2 = 3X$. Find the coordinates of the vertex and the focus.

(c) Prove that the sum of the reciprocals of two perpendicular focal chords of a conic is constant.

(d) Show that the ortho-centre of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my = 1$ is given by $\frac{x}{l} = \frac{y}{m} = \frac{a+b}{am^2 - 2hlm + bl^2}$

(e) Find the condition that the plane $lx + my + nz = p$ may touch the conicoid $ax^2 + by^2 + cz^2 = 1$. Verify that the plane $2x - 2y + 8z = 9$ touches the ellipsoid $x^2 + 2y^2 + 3z^2 = 9$.

(f) Show that the shortest distance between any two opposite edges of the tetrahedron formed by the planes $y + z = 0$, $z + x = 0$, $x + y = 0$, $x + y + z = a$ is $\frac{2a}{\sqrt{6}}$ and that the three lines of shortest distance intersect at the point $x = y = z = -a$.

(g) Find the equation to the cylinder generated by the lines drawn through the points of the circle

$x + y + z = 1, x^2 + y^2 + z^2 = 4$ which are

parallel to the line $\frac{x}{2} = \frac{y}{-1} = \frac{z}{2}$.

(h) A variable plane is parallel to the given plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$ and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the cone

$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0.$$