2016

MATHEMATICS

(Major)

Paper : 1.1

(Algebra and Trigonometry)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

1. Answer/Choose the correct option : 1×10=10

   (a) Let $A = \{1, 2, 3, 4\}$. Give an example of a relation in A which is transitive but not reflexive or symmetric.

   (b) Which statement is correct?

      (i) $f : R \to R$ given by $f(x) = x^2$ is injective.

      (ii) $f : N \to N$ given by $f(x) = 2x$ is surjective.

(iii) $f : N \to E$ given by $f(x) = 2x$ is not surjective. (E is the set of non-negative even integers)

(iv) $f : N \to N$ given by $f(x) = x^2$ is injective.

(c) If m, n and x are three elements of a group and $mnxnm = y$, then

(i) $x = n^{-1}m^{-1}ym^{-1}n^{-1}$

(ii) $x = nm^{-1}ym^{-1}n$

(iii) $x = m^{-1}n^{-1}yn^{-1}m^{-1}$

(iv) $x = m^{-1}n^{-1}ym^{-1}n^{-1}$

(d) For the group $\langle Z, +\rangle$ and normal subgroup $N = \{3n | n \in Z\}$, what is the order of the quotient group $\frac{Z}{N}$?

(e) If n is an integer, then $(1+i)^n + (1-i)^n$ equals

(i) $2^{\frac{n}{2}+1} \cos \frac{n\pi}{4}$

(ii) $2^{\frac{n}{2}} \cos \frac{n\pi}{4}$

(iii) $2^{\frac{2}{n}-1} \cos \frac{\pi}{4}$

(iv) $2^{\frac{n}{2}} \sin \frac{n\pi}{4}$

(f) Find out the correct statement(s) :

(i) $(1+\omega+\omega^2)^3 - (1-\omega+\omega^2)^3 = -1$

(ii) $(1+\omega+\omega^2)^3 - (1-\omega+\omega^2)^3 = 1$

(iii) $(1+\omega+\omega^2)^3 - (1-\omega+\omega^2)^3 = 0$

(iv) $(1+\omega+\omega^2)^3 - (1-\omega+\omega^2)^3 = \frac{i+\sqrt{3}}{2}$

(g) If α, β, γ, δ are the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$, then the value of $\Sigma\alpha^2\beta\gamma$ is

(i) $-pq+3r$

(ii) $pr-4s$

(iii) $q^2-2pr+2s$

(iv) $-r/s$

(h) The rank of the matrix

$$A = \begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{bmatrix}$$

is

(i) 1

(ii) 0

(iii) 2

(iv) 3

(i) If A is any square matrix, then find the correct statement(s) :

(i) $A+A'$ is not symmetric

(ii) $A-A'$ is symmetric

(iii) $A-A'$ is skew-symmetric

(iv) $A+A'$ is skew-symmetric

(j) A matrix is idempotent if $A^2 = A$. If $AB = A$ and $BA = B$, then show that A is idempotent.

2. Give answers to the following questions : 2×5=10

(a) With an example show that we can have maps f and g such that $g \circ f$ is one-one and onto but f need not be onto and g need not be one-one.

(b) Let A and B be two square matrices of same order. If $AB = I$, prove that $BA = I$.

(c) Find the centre of S_3 where $S = \{1, 2, 3\}$.

(d) If $\sin(\alpha + i\beta) = x + iy$, prove that

$$x^2 \operatorname{cosec}^2 \alpha - y^2 \sec^2 \alpha = 1$$

(e) How many complex roots does the equation $x^4 + 2x^2 + 3x - 1 = 0$ have? Apply Descartes' rule of signs for finding the complex roots.

3. Answer any *four* parts : 5×4=20

(a) Let $f : A \to B$ and $g : B \to C$ be one-to-one and onto functions. Show that $(g \circ f)^{-1}$ exists and $(g \circ f)^{-1} = f^{-1} \circ g^{-1} : C \to A$.

(b) If H and K are finite subgroups of G of order $O(H)$ and $O(K)$ respectively, then prove that

$$O(HK) = \frac{O(H)O(K)}{O(H \cap K)}$$

(c) Prove that a subgroup H of a group G is a normal subgroup of G if and only if the product of two right cosets is again a right coset of H in G.

(d) Deduce the following series :

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots + \frac{(-1)^{n-1} x^{2n-1}}{2n-1} + \cdots$$

(e) If α, β, γ be the roots of the equation $x^3 + px^2 + qx + r = 0$, then find the equation whose roots are $\alpha\beta + \beta\gamma$, $\beta\gamma + \gamma\alpha$, $\gamma\alpha + \alpha\beta$.

(f) Reduce the following matrix to normal form and find its rank :

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

4. Answer any *one* part : 10

(a) *(i)* Show that a subgroup of index 2 in a group G is a normal subgroup of G.

(ii) In a set of n elements define S_n and A_n where the symbols have their usual meanings.

(iii) Show by example that a quotient group may be Abelian but parent group of the quotient group may not be Abelian. 4+2+4

(b) *(i)* Show that an infinite cyclic group has precisely 2 generators.

(ii) Show that a group G of prime order cannot have non-trivial subgroups.

(iii) Let a, n $(n \geq 1)$ be any integers s.t. g.c.d. $(a, n) = 1$. Prove that

$$a^{\phi(n)} \equiv 1 (\text{mod}\, n)$$ 4+2+4

5. Answer any *one* part : 10

(a) *(i)* Show that the roots of the equation $Z^n = (Z+1)^n$ where n is a positive integer >1 are collinear points in the Z plane.

(ii) Using De Moivre's theorem, solve

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$$

(iii) If $\sin(\theta + i\phi) = \tan(x + iy)$, then show that $\dfrac{\tan\theta}{\tan\phi} = \dfrac{\sin 2x}{\sinh 2y}$. 3+4+3

(b) *(i)* Expand $\sin^4\theta\cos^2\theta$ in a series of cosines of multiples of θ.

(ii) If $x < (\sqrt{2} - 1)$, then prove that

$$2\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots\right) = \frac{2x}{1-x^2} - \frac{1}{3}\left(\frac{2x}{1-x^2}\right)^3 + \frac{1}{5}\left(\frac{2x}{1-x^2}\right)^5 - \cdots$$

(iii) Show that

$$\tan\left(i \log \frac{a - ib}{a + ib}\right) = \frac{2ab}{a^2 - b^2}$$ 3+4+3

6. Answer any *two* parts : 5×2=10

(a) If α, β, γ, δ are roots of the biquadratic equation $x^4 + px^3 + qx^2 + rx + \delta = 0$, find the value of $\Sigma\alpha^2\beta\gamma$ and $\Sigma\alpha^2\beta^2$.

(b) Solve by Cardan's method :

$$x^3 - 6x^2 - 6x - 7 = 0$$

(c) Find the equation whose roots are squares of the differences of the roots of the equation $x^3 + x + 2 = 0$ and deduce from the resulting equation the nature of the roots of the given cubic.

(d) Solve the equation

$$16x^3 - 44x^2 + 36x - 9 = 0,$$

given that roots are in harmonic progression.

7. Answer any *two* parts : 5×2=10

(a) Define skew Hermitian matrix. Prove that every Harmitian matrix can be written as $A = B + iC$, where B is real and symmetric and C is real and skew-symmetric.

(b) (i) If a non-singular matrix A is symmetric, prove that A^{-1} is also symmetric. 2

(ii) If A is a $n \times n$ non-singular matrix, then prove that $(A')^{-1} = (A^{-1})'$. 3

(c) Investigate for what values of a and b the simulteneous equations

$$x_1 + x_2 + x_3 = 6$$
$$x_1 + 2x_2 + 3x_3 = 10$$
$$x_1 + 2x_2 + ax_3 = b$$

have

(i) no solution;

(ii) an unique solution;

(iii) an infinite number of solutions.

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2016

MATHEMATICS

(Major)

Paper : 1.2

(Calculus)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following : 1×10=10

(a) Write down the nth derivative of $\cos(2x+3)$.

(b) If $z = x^3 y^5 \phi(x/y)$, find the value of

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

(c) Find the expression for the subnormal to the curve $y^2 = 4ax$ at any point $P(x, y)$ on the curve.

(d) Write down the radius of curvature for the curve $s = c\tan\psi$.

(e) Write down the asymptotes to the curve $xy = a^2$.

(f) If $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right) + \cot^{-1}\left(\frac{y}{x}\right)$, $x \neq 0$, find $\frac{\partial f}{\partial x}$.

(g) If $I = \int \sqrt{x^2 - a^2}\, dx$, write down the expression for I.

(h) Write down the value of $\int_0^\pi |\cos x| dx$.

(i) What is the volume of the solid generated due to the revolution of the circle $x^2 + y^2 = a^2$ about x-axis?

(j) Evaluate $\int_{-\pi}^{\pi} |x| \sin x\, dx$.

2. Answer the following questions : 2×5=10

(a) If $y = \sin x \sin 2x \sin 3x$, find y_n.

(b) Show the pedal equation of the curve $r = e^\theta$ is $2p^2 = r^2$.

(c) Prove that $\int_0^\pi x \cos^4 x\, dx = \frac{3\pi^2}{16}$.

(d) Show that the perimeter of the circle $x^2 + y^2 = a^2$ is $2\pi a$.

(e) Find the area of the region bounded by the parabola $y^2 = 4x$ and its latus rectum.

3. Answer the following : 5×4=20

(a) If $u = f(x, y)$ where $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

(b) Trace the curve $x^3 + y^3 = 3axy$.

Or

Prove that the sum of the intercepts of the tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ upon the coordinate axes is constant.

(c) Integrate $\int (\sqrt{\tan x} + \sqrt{\cot x})\, dx$.

Or

Integrate $\int \frac{dx}{(x^2 - 2x + 1)\sqrt{x^2 - 2x + 3}}$.

(d) Find the whole length of the loop of the curve $9y^2 = (x+7)(x+4)^2$.

4. Answer *either (a) or (b)* :

(a) *(i)* If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, $|x|<1$, show that

$$(1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2 y_n = 0$$ 6

(ii) If $y = x^{n-1}\log x$, show that $y_n = \frac{(n-1)!}{x}$. 4

(b) *(i)* If u is a homogeneous function of x and y of degree n, having continuous partial derivatives, prove that

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 u = n(n-1)u$$ 6

(ii) If $v = \sin^{-1}\frac{x^2+y^2}{x+y}$, then show that

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \tan v$$ 4

5. Answer *either (a) or (b)* :

(a) *(i)* Find the asymptotes of the curve $x^4 - x^2y^2 + x^2 + y^2 - a^2 = 0$. 5

(ii) Show that for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

the radius of curvature at an extremity of the major axis is equal to half the latus rectum. 5

(b) Define cusp, isolated points, single cusp and double cusp. Find the position and nature of the multiple points on the curve $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$. 4+6

6. *(a)* If $U_n = \int_0^{\pi/2} x^n \sin x \, dx \; (n \geq 1)$, show that

$$U_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)u_{n-2}$$ 5

(b) If $J_n = \int (a^2+x^2)^{n/2} dx$, show that

$$J_n = \frac{x(a^2+x^2)^{n/2}}{n+1} + \frac{na^2}{n+1}J_{n-2}$$ 5

7. *(a)* Show that the area enclosed by the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $\frac{3}{8}\pi a^2$. 5

(b) Find the surface area of the solid generated by revolving the cardioid $r = a(1+\cos\theta)$ about the initial line. 5

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