2025

MATHEMATICS

Paper : MAT0400404

(Number Theory–I)

Full Marks : 60

Time : 2½ hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : $1\times8=8$

   (a) State Well-Ordering Principle.

   (b) If a and b are integers with $b\neq0$, then there exist unique integers q and r such that $a = qb + r$ where

      (i) $0 < r \leq b$

      (ii) $0 \leq r < |b|$

      (iii) $0 \leq r \leq b$

      (iv) $0 \leq r \leq |b|$

   (Choose the correct option)

(c) Which of the following Diophantine equation cannot be solved ?

(i) $6x+51y=22$

(ii) $24x+138y=18$

(iii) $158x-57y=7$

(iv) $221x+35y=11$

(d) Give an example to show that $a^2 \equiv b^2 \pmod n$ need not imply that $a \equiv b \pmod n$.

(e) Without performing the division, determine whether the integer 176, 521, 221, is divisible by 9.

(f) If p is a prime number, then

(i) $(p-1)! \equiv 1 \pmod p$

(ii) $(p-1)! \equiv -1 \pmod p$

(iii) $(p+1)! \equiv 1 \pmod p$

(iv) $(p+1)! \equiv -1 \pmod p$

(Choose the correct option)

(g) Find $\sigma(180)$.

(h) Define Möbius μ–function.

2. Answer the following questions : $2\times6=12$

(a) If $a|c$ and $b|c$ with $gcd\ (a,b)=1$, then prove that $ab|c$.

(b) Prove that $gcd\ (a+b, a-b)=1$ or 2 if $gcd\ (a,b)=1$.

(c) Use Fermat's theorem to show that $5^{38} \equiv 4 \pmod{11}$.

(d) Show that 41 divides $2^{20}-1$.

(e) If n is a square free integer, prove that $\tau(n)=2^r$, where r is the number of prime divisors of n.

(f) For $n>2$, prove that $\phi(n)$ is an even integer.

3. Answer ***any four*** of the following questions : 5×4=20

(a) State and prove Archimedean property. 1+4=5

(b) Use the Euclidean Algorithm to obtain integers x and y satisfying

$gcd(12378, 3054) = 12378x + 3054y$

(c) Use Chinese Remainder Theorem to solve the simultaneous congruences

$$x \equiv 2(mod\,3)$$

$$x \equiv 3(mod\,5)$$

$$x \equiv 2(mod\,7)$$

(d) If n and r are positive integers with $1 \le r < n$, then prove that the bionomial coefficient

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

is also an integer.

(e) Prove that every positive integer $n>1$ can be expressed uniquely as a product of primes a part from the order in which the factor's occur.

(f) If p and q are distinct primes, prove that $p^{q-1} + q^{p-1} \equiv 1(mod\,pq)$

(g) If f is a multiplicative function and F be defined by $F(n) = \sum_{d|n} f(d)$, then prove that F is also multiplicative.

(h) If $n \ge 1$ and $gcd\,(a,n)=1$, then prove that $a^{\phi(n)} \equiv 1(mod\,n)$.

4. Answer ***any two*** of the following questions : 10×2=20

(a) (i) Prove that for given integers a and b, with $b>0$, there exist unique integers q and r satisfying

$a = bq + r,\ 0 \le r < b$ 6

(ii) Establish the following formula by Mathematical induction.

$$1{\cdot}2 + 2{\cdot}3 + 3{\cdot}4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

for all $n \ge 1$ 4

(b) (i) Given integers a and b, not both of which are zero, prove that there exist integers x and y such that $gcd\ (a,b) = ax + by$. 5

(ii) Determine all solutions in the positive integers of the Diophantine equation $172x + 20y = 1000$ 5

(c) (i) Prove that if p is a prime and $p \nmid a$, then

$a^{p-1} \equiv 1 (mod\ p)$

Is the converse of it true ? Justify. 5+1=6

(ii) Solve : $9x \equiv 21 (mod\ 30)$. 4

(d) (i) Prove that there is an infinite number of primes. 4

(ii) Prove that the quadratic congruence $x^2 + 1 \equiv 0 (mod\ p)$ where p is an odd prime has a solution if and only if $p \equiv 1 (mod\ 4)$. 6

(e) (i) If $n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$ is the prime factorization of $n > 1$, then prove that

(I) $\tau(n) = (k_1 + 1)(k_2 + 1) \ldots\ldots (k_r + 1)$

(II) $\sigma(n) = \left(\frac{p_1^{k_1+1} - 1}{p_1 - 1}\right)\left(\frac{p_2^{k_2+1} - 1}{p_2 - 1}\right) \ldots\ldots \left(\frac{p_r^{k_r+1} - 1}{p_r - 1}\right)$

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(ii) For each positive integer $n \geq 1$, prove that

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & if\ n = 1 \\ 0, & if\ n > 1 \end{cases}$$ 4