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**3 (Sem–5/CBCS) MAT HC 1**

**2024**

# MATHEMATICS

( Honours Core )

**( New Course )**

Paper : MAT–HC–5016

***( Complex Analysis )***

*Full Marks : 60*

Time : Three hours

***The figures in the margin indicate full marks for the questions.***



1. Answer the following questions : 1×7=7

(a) The function

$$f(z) = xy^2 + e^{xy} + i(2x - y) \text{ is}$$

continuous everywhere in the complex plane. *(State True **or** False)*

Contd.

(b) Let the function $f(z) = u(x,y) + iv(x,y)$ be defined throughout some ε-neighbourhood of a point $z_0 = x_0 + iy_0$. State a sufficient condition for existence of the derivative $f'(z_0)$.

(c) Find $\lim\limits_{z \to -1} \dfrac{iz+3}{z+1}$.

(d) Define entire function and give an example.

(e) Show that $exp(2 \pm 3\pi i) = -e^2$.

(f) What is a Jordan curve?

(g) State Liouville's theorem.

2. Answer the following questions : 2×4=8

(a) Using $\varepsilon - \delta$ definition, show that if $f(z) = z^2$ then $\lim\limits_{z \to z_0} f(z) = z_0^2$.

(b) Show that $f(z) = \begin{Bmatrix} z^2, z \neq z_0 \\ 0, z = z_0 \end{Bmatrix}$ where $z_0 \neq 0$ is discontinuous at $z = z_0$.





(c) Determine the singular points of the function, $f(z) = \dfrac{2z+1}{z(z^2+1)}$.

(d) Evaluate $\int\limits_C \bar{z}\,dz$ from $z = 0$ to $z = 4+2i$ along the curve $C$ given by $z = t^2 + it$.

3. Answer ***any three*** questions from the following : 5×3=15

(a) Show that the three cube roots of $-8i$ lie at the vertices of an equilateral triangle that is inscribed in a circle of radius 2 centred at the origin.

(b) Prove that $\lim\limits_{z \to 0} \dfrac{\bar{z}}{z}$ does not exist.

(c) If $f'(z) = 0$ everywhere in a domain $D$, then prove that $f(z)$ must be constant throughout $D$.

(d) Suppose that a function $f(z)$ is analytic at a point $z_0 = z(t_0)$ on a differentiable arc $z = z(t)(a \le t \le b)$. Show that if $w(t) = f(z(t))$ then $w'(t) = f'(z(t))z'(t)$ when $t = t_0$.

(e) Using anti-derivative, evaluate the integral $\int_C z^{1/2}dz$, where $C$ is a contour from $z=-3$ to $z=3$ that, except for its end points, lies above the $x$-axis.

4. Answer **any three** questions from the following : 10×3=30

(a) (i) If $z_0$ and $w_0$ are points in the $z$ and $w$ planes respectively, then prove that

(A) $\lim_{z \to z_0} f(z) = \infty$ if and only if

$$\lim_{z \to z_0} \frac{1}{f(z)} = 0$$ and

(B) $\lim_{z \to \infty} f(z) = w_0$ if and only if

$$\lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0$$ 5

(ii) Let $u$ and $v$ denote the real and imaginary components of the function $f$ defined by the equations :

$$f(z) = \begin{cases} \frac{(\bar{z})^2}{z}, & when\ z \neq 0 \\ 0, & when\ z = 0 \end{cases}$$

Verify the Cauchy-Riemann equations at the origin $z=(0,0)$. 5



(b) (i) Show that $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$. 2

(ii) Find numbers $z = x+iy$ such that $e^z = 1+i$. 3

(iii) Show that if a function $f(z) = u(x,y) + iv(x,y)$ and its conjugate $\overline{f(z)}$ are both analytic in a domain $D$, then $f(z)$ must be constant throughout $D$. 5

(c) (i) Show that the zeros of $\sin z$ are all real. 2

(ii) Evaluate $\int_0^{\pi/4} e^{it}\, dt$. 3

(iii) If $w = f(z) = \frac{1+z}{1-z}$ find $\frac{dw}{dz}$ and determine where $f(z)$ is not analytic. 5

 Contd.

(d) (i) If $w(t)$ is a piecewise continuous complex valued function defined on an interval $a \le t \le b$, then show that $\left|\int_a^b w(t)dt\right| \le \int_a^b |w(t)|\,dt$ 5

(ii) Let $C$ denote the line segment from $z = i$ to $z = 1$. By observing that of all the points on that line segment, the midpoint is the closest to the origin, show that $\left|\int_C \frac{dz}{z^4}\right| \le 4\sqrt{2}$, without evaluating the integral. 5

(e) (i) Show that

$$\int \frac{dz}{z^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{z}{a} + C_1 = \frac{1}{2ai}\ln\left(\frac{z - ai}{z + ai}\right) + C_2$$

5

(ii) Evaluate :

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}dz,$$ 5

where $C$ is the circle $|z| = 3$.



(f) (i) Prove that if a function $f$ is analytic at a given point, then its derivatives of all orders are also analytic at that point. 5

(ii) Let $C$ denote the positively oriented boundary of a square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate

$$\int_C \frac{\tan(z/2)}{(z - x_0)^2}dz \quad (-2 < x_0 < 2).$$ 5

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**3 (Sem–5/CBCS) MAT HC 2**

**2024**

**MATHEMATICS**

( Honours Core )

Paper : MAT–HC–5026

***( Linear Algebra )***

*Full Marks : 80*

Time : Three hours

***The figures in the margin indicate full marks for the questions.***

1. Answer the following questions as directed : $1\times10=10$

(a) Give reason why a line in $\mathbb{R}^2$ not passing through the origin is not a subspace of $\mathbb{R}^2$.

(b) Express $W = \left\{ \begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix}; a,b \in \mathbb{R} \right\}$

as span of two vectors.

*Contd.*

(c) State whether the fallowing statement is true *or* false :

"A finite dimensional vector space has exactly one basis."

(d) Find the dimension of the subspace of all vectors in $\mathbb{R}^3$ whose first and third entries are equal.

(e) 0 is an eigenvalue of a matrix $A$ if and only if $A$ is ______. *(Fill in the blank)*

(f) When is a square matrix said to be diagonalizable?

(g) Write the kernel of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(x,y,z) = (x,0,z)$.

(h) If $u = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$, then compute $u.v$ .

(i) What is the distance between the vectors $\vec{u} = (7,1)$ and $\vec{v} = (3,2)$ in the $\mathbb{R}^2$ plane ?



(j) What do you mean by orthogonal vectors in an inner product space?

2. Answer the following questions: 2×5=10

(a) Let $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Determine if $w$ is in null space of $A$.

(b) Let $\mathbb{P}_3$ be the vector space of all polynomials of degree atmost 3.

Are the vectors

$p(t) = 1 + t^2$ and $q(t) = 1 - t^2$ linearly independent in $\mathbb{P}_3$? Justify your answer.



(c) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 4 & -3 \end{bmatrix}$$

(d) Let $\mathcal{B} = \{b_1, b_2, b_3\}$ be a basis for a vector space $V$ and $T : V \to \mathbb{R}^2$ be a linear transformation such that

$$T(x_1 b_1 + x_2 b_2 + x_3 b_3) = \begin{bmatrix} 2x_1 - 4x_2 + 5x_3 \\ -x_2 + 3x_3 \end{bmatrix}.$$

Find the matrix for $T$ relative to $\mathcal{B}$.

(e) Let $v=(1,-2,2,0)$ be a vector in $\mathbb{R}^4$. Find a unit vector $u$ in the same direction as $v$.

3. Answer ***any four*** questions : 5×4=20

(a) If a vector space $V$ has a basis $\mathcal{B}=\{b_1,b_2,...,b_n\}$, then prove that any set in $V$ containing more than $n$ vectors must be linearly dependent.

(b) Let $b_1=\begin{bmatrix}1\\-3\end{bmatrix}, b_2=\begin{bmatrix}2\\-5\end{bmatrix}$ and $x=\begin{bmatrix}-2\\1\end{bmatrix}$. 3+2=5

(i) Show that the set $\mathcal{B}=\{b_1,b_2\}$ is a basis of $\mathbb{R}^2$

(ii) Find the coordinate vector $[x]$ of $x$ relative to $\mathcal{B}$.

(c) Given that 2 is an eigenvalue of the matrix $A=\begin{bmatrix}4&-1&6\\2&1&6\\2&-1&8\end{bmatrix}$

Find a basis for the corresponding eigenspace.

(d) Prove that an $n\times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.



(e) Compute the orthogonal projection of $\begin{bmatrix}1\\7\end{bmatrix}$ onto the line through $\begin{bmatrix}-4\\2\end{bmatrix}$ and the origin.

(f) If $\{u,v\}$ is an orthonormal set in an inner product space $V$, then show that $\|u-v\|=\sqrt{2}$.

Answer either *(a)* **or** *(b)* from each of the following questions : 10×4=40

4. (a) Find the rank and the nullity of the matrix 10

$$A=\begin{bmatrix}-2&-5&8&0&-17\\1&3&-5&1&5\\3&11&-19&7&1\\1&7&-13&5&-3\end{bmatrix}$$

(b) Let $b_1=\begin{bmatrix}-9\\1\end{bmatrix}$, $b_2=\begin{bmatrix}-5\\-1\end{bmatrix}$, $c_1=\begin{bmatrix}1\\-4\end{bmatrix}$ and consider the bases for $\mathbb{R}^2$ given by $\mathcal{B}=\{b_1,b_2\}$ and $C=\{c_1,c_2\}$ 5+5=10

(i) Find the change-of-coordinates matrix from $C$ to $\mathcal{B}$

(ii) Find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$

5. (a) (i) If $n \times n$ matrices $A$ and $B$ are similar, then show that they have the same characteristic polynomial. 3

(ii) If $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ then find an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$ 7

(b) (i) Let $\mathcal{B} = \{b_1, b_2, b_3\}$ and $\mathcal{D} = \{d_1, d_2\}$ be bases for vector spaces $V$ and $W$ respectively. Let $T : V \to W$ be a linear transformation with the property that

$$T(b_1) = 3d_1 - 5d_2,\ T(b_2) = -d_1 + 6d_2,\ T(b_3) = 4d_2.$$

Find the matrix for $T$ relative to $\mathcal{B}$ and $\mathcal{D}$. 5



(ii) Let $A$ be a real $2 \times 2$ matrix with a complex eigenvalue $\lambda = a - bi\,(b \neq 0)$ and an associated eigenvector $v$ in $\mathbb{C}^2$. Show that

$$A(\mathrm{Re}\,v) = a\,\mathrm{Re}\,v + b\,\mathrm{Im}\,v \text{ and}$$

$$A(\mathrm{Im}\,v) = b\,\mathrm{Re}\,v + a\,\mathrm{Im}\,v$$ 5

6. (a) Let, $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right\}$ be a basis for a subspace $W$ of $\mathbb{R}^3$. Using the Gram-Schmidt process construct an orthogonal basis for $W$.
Hence, find an orthonormal basis.

8+2=10

(b) What do you mean by an inner product on a vector space V? Consider the inner product in $\mathbb{R}^2$ defined by $\langle u,v\rangle = 4u_1v_1 + 5u_2v_2$ where $u=(u_1,u_2),\ v=(v_1,v_2)\in\mathbb{R}^2$. If $x=(1,1)$ and $y=(5,-1)$, then find $\|x\|, \|y\|$ and $|\langle x,y\rangle|^2$. Also, show that in an inner product space $V$ over $\mathbb{R}$,

$$\langle u,v\rangle = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2,\ \forall u,v\in V.$$

2+3+5=10

7. (a) State Cayley-Hamilton theorem for matrices. Verify the theorem for the matrix. 2+6+2=10

$$M=\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

and hence find $M^{-1}$.

(b) If possible, diagonalize the symmetric matrix 10

$$A=\begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$