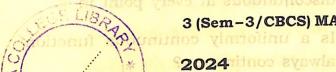
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3 (Sem-3/CBCS) MAT HC 1

## MATHEMATICS

(Honours Core)

Paper: MAT-HC-3016

# (Theory of Real Functions)

Full Marks: 80

Time: Three hours

# The figures in the margin indicate full marks for the questions.

- Answer the following questions: 1×10=10
  - Does  $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$  exist? (a)
  - Define a cluster point of a set  $S \subseteq \mathbb{R}$ . (b)
  - "If  $A \subseteq \mathbb{R}$  and  $\phi: A \to \mathbb{R}$  has a limit at (c) a point  $a \in \mathbb{R}$ , then  $\phi$  is bounded on some neighbourhood of a." Mention the truth or falsity of this statement.

- (d) Give an example of a function which is discontinuous at every point in  $\mathbb{R}$ .
- (e) Is a uniformly continuous function always continuous?
- (f) Mention the points of discontinuity of the greatest integer function f(x) = [x].
- (g) Is a function continuous at a point always differentiable at that point?
- (h) State Darboux's theorem.
- (i) Write Taylor's series for a function f, defined on an interval I, about a point  $a \in I$  when f has all orders of derivatives at a.
- (j) Write the fourth term in the power series expansion of cosx.
- 2. Answer the following questions: 2×5=10
  - (a) Show that  $\lim_{x\to a} x^3 = a^3$  by using the  $\varepsilon \delta$  definition of limit.

- (b) Prove that a constant function is continuous everywhere.
- (c) Applying sequential criterion for limit establish that  $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$ .
- (d) Find the points of discontinuity of the function  $f(x) = \frac{(x-3)(x^2+1)}{(x+2)(x-4)}$ .
  - (e) Evaluate the limit  $\lim_{x\to\infty} \frac{\sqrt{x}-x}{\sqrt{x}+x}$ , if it exists.
- 3. Answer **any four** parts of the following: 5×4=20
  - (a) If  $f: D \to \mathbb{R}$  and a is a cluster point of D, then prove that f can have only one limit at a if the limit exists.
    - If  $f: I \to \mathbb{R}$ , where I = [a, b] be a closed bounded interval, is continuous on I, then prove that f has an absolute maximum and an absolute minimum on I.
  - (c) State and prove Bolzano's intermediate value theorem. 1+4=5

- (d) If I is a closed and bounded interval and  $f: I \to \mathbb{R}$  is continuous on I, then prove that f is uniformly continuous on I.
- (e) State Rolle's theorem and prove it.

  1+4=5
- (f) Determine whether x = 0 is a point of relative extremum of the function  $f(x) = \sin x x$ .
- 4. Answer **any four** parts of the following questions:
  - (a) If I = [a, b],  $f: I \to \mathbb{R}$  is continuous on I and if f(a) < 0 < f(b) or f(a) > 0 > f(b), then prove that there exists a number  $c \in (a, b)$  such that f(c) = 0.
    - (b) (i) If I = [a, b] be a closed bounded interval and  $f: I \to \mathbb{R}$  is continuous on I, then show that f is bounded on I.

(ii) Let P(x) be a polynomial function of degree n. Prove that

$$\lim_{x \to a} P_n(x) = P_n(a).$$
 5

- (c) (i) If a function f is uniformly continuous on a bounded subset A of  $\mathbb{R}$ , then prove that f is bounded on A.
- (ii) Show that the function  $f(x) = \frac{1}{x}$  is uniformly continuous on  $I = [1, \infty)$ .
- (d) (i) If K > 0 and the function  $f: \mathbb{R} \to \mathbb{R}$  satisfies the condition  $|f(x)-f(y)| \le K|x-y|$ , for all real numbers x and y, then show that f is continuous at every point  $c \in \mathbb{R}$ . Further, from it conclude that f(x) = |x| is continuous at every point  $c \in \mathbb{R}$ . 4+2=6



(ii) Show that the function f defined by

$$f(x) = \frac{e^{1/x} - 1}{e^{1/x} + 1}$$
, if  $x \neq 0$ 

$$(0) = 1$$
 is uniformly continuous on a bounded subset

at x = 0.

- (e) State Caratheodory's theorem and prove it completely. Apply this theorem to show that  $f(x) = 2x^3 + 1$  is differentiable at  $a \in \mathbb{R}$  and that  $f'(a) = 6a^2$ . 2+4+4=10
- (f) If  $f: I \to \mathbb{R}$  is differentiable on the interval I, then prove that
- 10 (i) f is increasing iff  $f'(x) \ge 0$ ,  $\forall x \in I$ .
- (ii) f is decreasing iff  $f'(x) \le 0$ ,  $\forall x \in I$ . Hence prove that

$$f(x) = x^3 - \frac{9}{2}x^2 + 6x - 1$$

is decreasing in the interval (1,2).  $3\frac{1}{2}+3\frac{1}{2}+3=10$ 

- (g) (i) Find the derivative of  $f(x) = \sin \sqrt{x} \text{ using the definition of derivative.}$ 
  - (ii) State and prove Cauchy's Mean Value Theorem. 2+4=6
- (h) (i) Evaluate:  $\lim_{x\to 0} \frac{x^2 \sin^2 x}{x^4}$ . 5
  - (ii) Prove that  $e^{\pi} > \pi^e$ . 5



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## 3 (Sem-3/CBCS) MAT HC 2

#### 2024

## MATHEMATICS

(Honours Core)

Paper: MAT-HC-3026

(Group Theory-I)

Full Marks: 80

Time: Three hours

# The figures in the margin indicate full marks for the questions.

- 1. Answer the following questions as directed: 1×10=10
  - (a) "The set S of positive irrational numbers together with 1 is a group under multiplication." Justify whether it is true or false.
  - (b) "In the set of integers, subtraction is not associative." Justify the statement.
  - (c) "Product of two subgroups of a group is again a subgroup." State whether true or false.

- (d) In the group  $\mathbb{Z}_{12}$ , find the order of 6.
- (e) Write all the generators of  $\mathbb{Z}_8$ .
- (f) List all the elements of the group  $\frac{\mathbb{Z}}{4\mathbb{Z}}$ .
- (g) Give the statement of Cayley's theorem.
- (h) Write the following permutation as product of 2-cycles:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 4 & 6 & 5 & 7 & 2 \end{pmatrix}$$

- (i) State whether the following statement is true or false:

  "If the homomorphic image of a group is Abelian, then the group itself is Abelian."
- (j) Give the statement of second isomorphism theorem.
- 2. Answer the following questions: 2×5=10
  - (a) Prove that in a group G, for any elements a and b and any integer n,  $(a^{-1}ba)^n = a^{-1}b^na.$

- (b) Show that in a group G, right and left cancellation laws hold.
- (c) Define centre of a group G and give an example.
- (d) What is meant by cycle of a permutation? Give an example.
- (e) If  $\phi$  is a homomorphism from a group G onto a group  $\overline{G}$ , then show that  $\phi$  carries the identity element of G to the identity element of  $\overline{G}$ .
- 3. Answer any four questions: 5×4=20
  - (a) Let G be a group and H be a nonempty finite subset of G. Prove that H is a subgroup of G if and only if H is closed under the operation in G.
  - (b) Let G be a group and H be a subgroup of G. For an element a in G, prove that
     aH = H if and only if a is in H.
  - (c) Let G be a finite group and H be a subgroup of G. Prove that |H| divides |G|.

- (d) Prove that in a finite group, the number of elements of order d is divisible by  $\phi(d)$ .
- (e) Define external direct product of a finite collection of groups. List all elements of  $U(8) \oplus U(10)$  and find  $|U(8) \oplus U(10)|$ .

  2+2+1=5
  - (f) Let  $\phi$  be a homomorphism from a group G to a group  $\overline{G}$ . If  $\overline{K}$  is a normal subgroup of  $\overline{G}$ , prove that  $\phi^{-1}(\overline{K}) = \{k \in G \mid \phi(k) \in \overline{K}\}$  is a normal subgroup of G.

Answer either (a) or (b) from each of the following questions (Q.4 to Q.7):  $10\times4=40$ 

4. (a) Describe the elements of  $D_4$ , the symmetrics of a square. Write down a complete Cayley's table for  $D_4$ . Show that  $D_4$  forms a group under composition of functions. Is  $D_4$  an Abelian group?

2+3+4+1=10

- (b) Let G be a group and H be a non-empty subset of H. Prove that H is a subgroup of G if and only if  $a.b^{-1}$  is in H whenever a and b are in H. Also, write all the subgroups of the group of integers  $\mathbb{Z}$ . 8+2=10
- 5. (a) (i) Let a be an element of order n in a group G and let k be a positive integer. Then prove that

 $\langle a^k \rangle = \langle a^{gcd(n,k)} \rangle$  and



$$\left|a^{k}\right| = \frac{n}{\gcd(n,k)}$$

Prove that in a finite cyclic group, the order of an element divides the order of the group. 8+2=10

(b) Prove that every subgroup of a cyclic group is cyclic. Also, if  $|\langle a \rangle| = n$ , then show that the order of any subgroup of  $\langle a \rangle$  is a divisor of n; and for each positive divisor k of n, the group  $\langle a \rangle$  has exactly one subgroup of order k, which is  $\langle a^{n/k} \rangle$ . 5+2+3=10

- 6. (a) Prove that every group is isomorphic to a group of permutations.
  - (b) Prove that the order of a permutation of a finite set written in disjoint cycle form is the least common multiple of the lengths of the cycles.
- 7. (a) (i) Let  $\phi$  be a group homomorphism from a group G to a group  $\overline{G}$ .

  Prove that  $\frac{G}{Ker \phi}$  is isomorphic to  $\phi(G)$ .

group silve stim and 
$$\frac{Z}{\langle n \rangle} \cong Z_n$$
.  $7+3=10$ 

- (b) Let  $\phi$  be an isomorphism from a group G onto a group  $\overline{G}$ . Prove that:
- for every integer n and for every group element a in G,  $\phi(a^n) = [\phi(a)]^n.$
- (ii)  $|a| = |\phi(a)|$  for all a in G.

(iii) if G is finite, then G and  $\overline{G}$  have exactly the same number of elements of every order.

4+3+3=10



3 (Sem-3/CBCS) MAT HC 3



#### 2024

#### MATHEMATICS

(Honours Core)

Paper: MAT-HC-3036

(Analytical Geometry)

Full Marks: 80

Time: Three hours

The figures in the margin indicate full marks for the questions.

1. Answer all the questions:

1×10=10

- (a) Find the equation to the locus of the point P(t, 2t), where t is a parameter.
  - (b) Find the eccentricity of the hyperbola  $x^2 y^2 = 1$ .
  - (c) Find the angle between the pair of lines  $x^2 y^2 = 0$ .

- (d) What is the polar equation of a circle with the pole as the centre?
- (e) Under what condition does the equation  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  represent a pair of parallel straight lines?
- (f) Write down the equation of the z-axis in symmetric form.
- (g) What are the direction cosines of the normal to the plane 2x-y+2z=3?
- (h) Find the equation of the cone whose vertex is the origin and the guiding curve is x = a,  $y^2 + z^2 = b^2$ .
- (i) Define the shortest distance between two skew lines.
- (i) For what value of a, the transformation x' = -x + 2, y' = ay + 3 is a translation?
- 2. Answer all the questions: 2×5=10
  - (a) Find the value of k, if the equation kxy 8x + 9y 12 = 0 represents a pair of straight lines.

- (b) If the axes are rotated through an angle  $tan^{-1}2$ , what does the equation  $4xy-3x^2=a^2$  become?
- (c) The axes of a right circular cylinder is  $\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z-3}{2}$  and the radius is 5. Find the equation of the cylinder.
- (d) If  $e_1$  and  $e_2$  are the eccentricities of a hyperbola and its conjugate, show that  $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$ 
  - (e) Find the equation of the sphere passing through the circles  $x^2 + y^2 + z^2 = 9$ , 2x + 3y + 4z = 5 and the point (1, 2, 3).
- 3. Answer any four questions: 5×4=20
  - (a) If by rotation of axes about the origin, the expression  $ax^2 + 2hxy + by^2$  changes to  $a'x'^2 + 2h'x'y' + b'y'^2$ , then prove that a + b = a' + b' and  $ab h^2 = a'b' h'^2$ .
    - (b) Deduce the polar equation of a conic with the focus as the pole.

- (c) Find the equation of the tangent to the hyperbola  $4x^2 9y^2 = 1$  which is parallel to the line 4y = 5x + 7.
- (d) Prove that the equation  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  represents a pair of parallel straight lines if  $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$ .
- (e) Show that the equation of the cone whose vertex is the origin and the guiding curve is z = k, f(x, y) = 0, is

$$f\left(\frac{kx}{z},\frac{ky}{z}\right)=0.$$

(f) Find the equation of the director sphere of the conicoid  $ax^2 + by^2 + cz^2 = 1$ .

Answer either (a) or (b) from the following questions: 10×4=40

4. (a) (i) If  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  represents a pair of straight lines equidistant from the origin, then show that

$$f^4 - g^4 = c \left( b f^2 - a g^2 \right).$$

(ii) Find the lengths of semi-axes of some local conic  $ax^2 + 2hxy + by^2 = 1$ .

5+5=10

- (b) (i) Find the asymptotes of the hyperbola xy + ax + by = 0.
  - (ii) Reduce the equation  $7x^2 2xy + 7y^2 16x + 16y 8 = 0$ to the standard form. 5+5=10
- (a) (i) Show that the line lx + my = n is a tangent to the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ,
  - (ii) Show that the locus of the points of intersection of perpendicular is its directrix. 5+5=10
  - (b) (i) If the chord PP' of a hyperbola meets the asymptotes at Q and Q', then show that PQ = P'Q'.

5.

(ii) If PSP' and QSQ' are two .1 = Sud perpendicular focal chords of a conic, prove that

$$\frac{1}{PS.SP'} + \frac{1}{OS.SO'} = a \text{ (constant)}.$$

0 = yd + xx + yx aloched 5+5=10

(a) (i) Deduce the expression of the 0 = 8 - val shortest distance between the skew lines to the standard form.

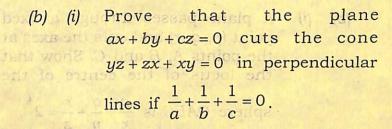
$$\frac{x-\alpha}{s_{non}} = \frac{y-\beta}{s_{non}} = \frac{z-\gamma}{s_{non}}$$
(i)

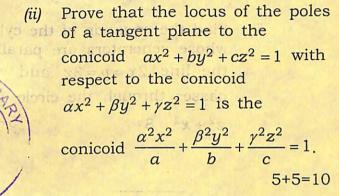
$$I = \frac{s_{y}}{s_{z}} + \frac{s_{z}}{s_{z}} = \frac{x - \alpha'}{m'} = \frac{z - \gamma'}{n's}.$$

A variable plane is parallel to the ziniog on legiven plane  $\frac{x}{a} + \frac{y}{b} + \frac{z}{a} = 0$  and meets the axes at the points A, B, C respectively. Prove that the circle ABC lies on the cone

$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0.$$

$$5+5=10$$





Show that the director sphere of the ellipsoid  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1$  is the sphere  $x^2 + u^2 + z^2 = a^2 + b^2 + c^2$ .

Obtain the equation of the chord of the conic  $\frac{1}{r} = 1 + e \cos \theta$ , joining the two points on the conic, whose vectorial angles are  $(\alpha + \beta)$  and  $(\alpha - \beta)$ . 5+5=10

(a)

(i)

(b) (i) A plane passes through a fixed point (p, q, r) and cuts the axes at the points A, B and C. Show that the locus of the centre of the

sphere OABC is 
$$\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 2$$
.

(ii) Find the equation of the cylinder whose generators are parallel to the line 2x = y = 3z and which passes through the circle y = 0,  $z^2 + x^2 = 8$ . 5+5=10



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Show that the director sphere of