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3 (Sem–3/CBCS) MAT HC 1

2024

MATHEMATICS

(Honours Core)

Paper : MAT–HC–3016

(Theory of Real Functions)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×10=10

(a) Does $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$ exist ?

(b) Define a cluster point of a set $S \subseteq \mathbb{R}$.

(c) "If $A \subseteq \mathbb{R}$ and $\phi : A \to \mathbb{R}$ has a limit at a point $a \in \mathbb{R}$, then ϕ is bounded on some neighbourhood of a." Mention the truth or falsity of this statement.

Contd.

(d) Give an example of a function which is discontinuous at every point in $\mathbb{R}$.

(e) Is a uniformly continuous function always continuous?

(f) Mention the points of discontinuity of the greatest integer function $f(x)=[x]$.

(g) Is a function continuous at a point always differentiable at that point?

(h) State Darboux's theorem.

(i) Write Taylor's series for a function f, defined on an interval I, about a point $a \in I$ when f has all orders of derivatives at a.

(j) Write the fourth term in the power series expansion of $\cos x$.

2. Answer the following questions: 2×5=10

(a) Show that $\lim_{x\to a} x^3 = a^3$ by using the $\varepsilon-\delta$ definition of limit.

(b) Prove that a constant function is continuous everywhere.

(c) Applying sequential criterion for limit establish that $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)=0$.

(d) Find the points of discontinuity of the function $f(x)=\dfrac{(x-3)(x^2+1)}{(x+2)(x-4)}$.

(e) Evaluate the limit $\lim_{x\to\infty} \dfrac{\sqrt{x}-x}{\sqrt{x}+x}$, if it exists.

3. Answer **any four** parts of the following: 5×4=20

(a) If $f: D \to \mathbb{R}$ and a is a cluster point of D, then prove that f can have only one limit at a if the limit exists.

(b) If $f: I \to \mathbb{R}$, where $I=[a,b]$ be a closed bounded interval, is continuous on I, then prove that f has an absolute maximum and an absolute minimum on I.

(c) State and prove Bolzano's intermediate value theorem. 1+4=5

(d) If I is a closed and bounded interval and $f : I \to \mathbb{R}$ is continuous on I, then prove that f is uniformly continuous on I.

(e) State Rolle's theorem and prove it. 1+4=5

(f) Determine whether $x = 0$ is a point of relative extremum of the function

$f(x) = \sin x - x$.

4. Answer ***any four*** parts of the following questions : 10×4=40

(a) If $I = [a, b]$, $f : I \to \mathbb{R}$ is continuous on I and if $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$, then prove that there exists a number $c \in (a, b)$ such that $f(c) = 0$.

(b) (i) If $I = [a, b]$ be a closed bounded interval and $f : I \to \mathbb{R}$ is continuous on I, then show that f is bounded on I. 5

(ii) Let $P(x)$ be a polynomial function of degree n. Prove that

$$\lim_{x \to a} P_n(x) = P_n(a).$$ 5

(c) (i) If a function f is uniformly continuous on a bounded subset A of $\mathbb{R}$, then prove that f is bounded on A. 5

(ii) Show that the function $f(x) = 1/x$ is uniformly continuous on $I = [1, \infty)$.

(d) (i) If $K > 0$ and the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition $|f(x) - f(y)| \le K|x - y|$, for all real numbers x and y, then show that f is continuous at every point $c \in \mathbb{R}$. Further, from it conclude that $f(x) = |x|$ is continuous at every point $c \in \mathbb{R}$. 4+2=6 *(ii)* Show that the function f defined by

$$f(x) = \frac{e^{1/x} - 1}{e^{1/x} + 1}, \text{ if } x \neq 0$$

$$= 0 \quad , \text{ if } x = 0$$

is discontinuous at $x = 0$. 4

(e) State Caratheodory's theorem and prove it completely. Apply this theorem to show that $f(x) = 2x^3 + 1$ is differentiable at $a \in \mathbb{R}$ and that $f'(a) = 6a^2$. 2+4+4=10

(f) If $f : I \to \mathbb{R}$ is differentiable on the interval I, then prove that

(i) f is increasing iff $f'(x) \geq 0, \forall x \in I$.

(ii) f is decreasing iff $f'(x) \leq 0, \forall x \in I$.

Hence prove that

$$f(x) = x^3 - \frac{9}{2}x^2 + 6x - 1$$

is decreasing in the interval $(1, 2)$.

3½+3½+3=10

(g) *(i)* Find the derivative of $f(x) = \sin\sqrt{x}$ using the definition of derivative. 4

(ii) State and prove Cauchy's Mean Value Theorem. 2+4=6

(h) *(i)* Evaluate : $\lim_{x \to 0} \frac{x^2 - \sin^2 x}{x^4}$. 5

(ii) Prove that $e^\pi > \pi^e$. 5

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3 (Sem–3/CBCS) MAT HC 2

2024

MATHEMATICS

(Honours Core)

Paper : MAT–HC–3026

(Group Theory–I)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : $1\times10=10$

   (a) "The set S of positive irrational numbers together with 1 is a group under multiplication." Justify whether it is true or false.

   (b) "In the set of integers, subtraction is not associative." Justify the statement.

   (c) "Product of two subgroups of a group is again a subgroup." State whether true or false.

Contd.

(d) In the group $\mathbb{Z}_{12}$, find the order of 6.

(e) Write all the generators of $\mathbb{Z}_8$.

(f) List all the elements of the group $\frac{\mathbb{Z}}{4\mathbb{Z}}$.

(g) Give the statement of Cayley's theorem.

(h) Write the following permutation as product of 2-cycles :

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 4 & 6 & 5 & 7 & 2 \end{pmatrix}$$

(i) State whether the following statement is true or false :

"If the homomorphic image of a group is Abelian, then the group itself is Abelian."

(j) Give the statement of second isomorphism theorem.

2. Answer the following questions : 2×5=10

(a) Prove that in a group G, for any elements a and b and any integer n, $(a^{-1}ba)^n = a^{-1}b^n a$.

(b) Show that in a group G, right and left cancellation laws hold.

(c) Define centre of a group G and give an example.

(d) What is meant by cycle of a permutation ? Give an example.

(e) If ϕ is a homomorphism from a group G onto a group $\bar{G}$, then show that ϕ carries the identity element of G to the identity element of $\bar{G}$.

3. Answer **any four** questions : 5×4=20

(a) Let G be a group and H be a non-empty finite subset of G. Prove that H is a subgroup of G if and only if H is closed under the operation in G.

(b) Let G be a group and H be a subgroup of G. For an element a in G, prove that $aH = H$ if and only if a is in H.

(c) Let G be a finite group and H be a subgroup of G. Prove that $|H|$ divides $|G|$.

(d) Prove that in a finite group, the number of elements of order d is divisible by $\phi(d)$.

(e) Define external direct product of a finite collection of groups. List all elements of $U(8) \oplus U(10)$ and find $|U(8) \oplus U(10)|$. 2+2+1=5

(f) Let ϕ be a homomorphism from a group G to a group $\bar{G}$. If $\bar{K}$ is a normal subgroup of $\bar{G}$, prove that $\phi^{-1}(\bar{K}) = \{k \in G \mid \phi(k) \in \bar{K}\}$ is a normal subgroup of G.

Answer ***either*** *(a)* ***or*** *(b) from each of the following questions (Q.4 to Q.7) :* 10×4=40

4. (a) Describe the elements of D_4, the symmetrics of a square. Write down a complete Cayley's table for D_4. Show that D_4 forms a group under composition of functions. Is D_4 an Abelian group? 2+3+4+1=10

(b) Let G be a group and H be a non-empty subset of H. Prove that H is a subgroup of G if and only if $a.b^{-1}$ is in H whenever a and b are in H. Also, write all the subgroups of the group of integers $\mathbb{Z}$. 8+2=10

5. (a) (i) Let a be an element of order n in a group G and let k be a positive integer. Then prove that

$$\langle a^k \rangle = \langle a^{gcd(n,k)} \rangle \text{ and}$$

$$|a^k| = \frac{n}{gcd(n,k)}$$

(ii) Prove that in a finite cyclic group, the order of an element divides the order of the group. 8+2=10

(b) Prove that every subgroup of a cyclic group is cyclic. Also, if $|\langle a \rangle| = n$, then show that the order of any subgroup of $\langle a \rangle$ is a divisor of n ; and for each positive divisor k of n, the group $\langle a \rangle$ has exactly one subgroup of order k, which is $\langle a^{n/k} \rangle$. 5+2+3=10

6. (a) Prove that every group is isomorphic to a group of permutations.

(b) Prove that the order of a permutation of a finite set written in disjoint cycle form is the least common multiple of the lengths of the cycles.

7. (a) (i) Let ϕ be a group homomorphism from a group G to a group $\overline{G}$.

Prove that $\frac{G}{Ker\,\phi}$ is isomorphic to $\phi(G)$.

(ii) Prove that $\frac{\mathbb{Z}}{\langle n \rangle} \cong \mathbb{Z}_n$. 7+3=10

(b) Let ϕ be an isomorphism from a group G onto a group $\overline{G}$. Prove that :

(i) for every integer n and for every group element a in G, $\phi(a^n) = [\phi(a)]^n$.

(ii) $|a| = |\phi(a)|$ for all a in G.

(iii) if G is finite, then G and $\overline{G}$ have exactly the same number of elements of every order.

4+3+3=10

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3 (Sem–3/CBCS) MAT HC 3

2024

MATHEMATICS

(Honours Core)

Paper : MAT–HC–3036

(Analytical Geometry)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***all*** the questions : 1×10=10

(a) Find the equation to the locus of the point $P(t, 2t)$, where t is a parameter.

(b) Find the eccentricity of the hyperbola $x^2 - y^2 = 1$.

(c) Find the angle between the pair of lines $x^2 - y^2 = 0$.

Contd.

(d) What is the polar equation of a circle with the pole as the centre?

(e) Under what condition does the equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

represent a pair of parallel straight lines?

(f) Write down the equation of the z-axis in symmetric form.

(g) What are the direction cosines of the normal to the plane $2x - y + 2z = 3$?

(h) Find the equation of the cone whose vertex is the origin and the guiding curve is $x = a,\ y^2 + z^2 = b^2$.

(i) Define the shortest distance between *two* skew lines.

(j) For what value of a, the transformation $x' = -x + 2,\ y' = ay + 3$ is a translation?

2. Answer **all** the questions: 2×5=10

(a) Find the value of k, if the equation $kxy - 8x + 9y - 12 = 0$ represents a pair of straight lines.

(b) If the axes are rotated through an angle $tan^{-1}2$, what does the equation $4xy - 3x^2 = a^2$ become?

(c) The axes of a right circular cylinder is $\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z-3}{2}$ and the radius is 5. Find the equation of the cylinder.

(d) If e_1 and e_2 are the eccentricities of a hyperbola and its conjugate, show that

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1.$$

(e) Find the equation of the sphere passing through the circles $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ and the point (1, 2, 3).

3. Answer **any four** questions: 5×4=20

(a) If by rotation of axes about the origin, the expression $ax^2 + 2hxy + by^2$ changes to $a'x'^2 + 2h'x'y' + b'y'^2$, then prove that $a + b = a' + b'$ and $ab - h^2 = a'b' - h'^2$.

(b) Deduce the polar equation of a conic with the focus as the pole.

Contd.

(c) Find the equation of the tangent to the hyperbola $4x^2 - 9y^2 = 1$ which is parallel to the line $4y = 5x + 7$.

(d) Prove that the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a pair of parallel straight lines if $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$.

(e) Show that the equation of the cone whose vertex is the origin and the guiding curve is $z = k$, $f(x, y) = 0$, is

$$f\left(\frac{kx}{z}, \frac{ky}{z}\right) = 0.$$

(f) Find the equation of the director sphere of the conicoid $ax^2 + by^2 + cz^2 = 1$.

Answer ***either*** *(a)* ***or*** *(b) from the following questions:* 10×4=40

4\. (a) (i) If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines equidistant from the origin, then show that

$$f^4 - g^4 = c(bf^2 - ag^2).$$

(ii) Find the lengths of semi-axes of the conic $ax^2 + 2hxy + by^2 = 1$. 5+5=10

(b) (i) Find the asymptotes of the hyperbola $xy + ax + by = 0$.

(ii) Reduce the equation

$$7x^2 - 2xy + 7y^2 - 16x + 16y - 8 = 0$$

to the standard form. 5+5=10

5\. (a) (i) Show that the line $lx + my = n$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $a^2l^2 + b^2m^2 = n^2$.

(ii) Show that the locus of the points of intersection of perpendicular is its directrix. 5+5=10

(b) (i) If the chord PP' of a hyperbola meets the asymptotes at Q and Q', then show that $PQ = P'Q'$.

(ii) If PSP' and QSQ' are two perpendicular focal chords of a conic, prove that

$$\frac{1}{PS.SP'} + \frac{1}{QS.SQ'} = a \text{ (constant)}.$$

5+5=10

6. (a) (i) Deduce the expression of the shortest distance between the skew lines

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n},$$

$$\frac{x-\alpha'}{l'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}.$$

(ii) A variable plane is parallel to the given plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ and meets the axes at the points A, B, C respectively. Prove that the circle ABC lies on the cone

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

5+5=10

(b) (i) Prove that the plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

(ii) Prove that the locus of the poles of a tangent plane to the conicoid $ax^2+by^2+cz^2=1$ with respect to the conicoid $\alpha x^2+\beta y^2+\gamma z^2=1$ is the conicoid $\frac{\alpha^2x^2}{a}+\frac{\beta^2y^2}{b}+\frac{\gamma^2z^2}{c}=1$.

5+5=10

7. (a) (i) Show that the director sphere of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is the sphere $x^2+y^2+z^2=a^2+b^2+c^2$.

(ii) Obtain the equation of the chord of the conic $\frac{l}{r}=1+e\cos\theta$, joining the two points on the conic, whose vectorial angles are $(\alpha+\beta)$ and $(\alpha-\beta)$.

5+5=10

Contd.

(b) (i) A plane passes through a fixed point (p, q, r) and cuts the axes at the points A, B and C. Show that the locus of the centre of the sphere $OABC$ is $\frac{p}{x}+\frac{q}{y}+\frac{r}{z}=2$.

(ii) Find the equation of the cylinder whose generators are parallel to the line $2x = y = 3z$ and which passes through the circle $y = 0$, $z^2 + x^2 = 8$. 5+5=10
