Total number of printed pages–11

3 (Sem–4/CBCS) MAT HC 1

2024

MATHEMATICS

(Honours Core)

Paper : MAT–HC–4016

(Multivariate Calculus)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : $1\times10=10$

(a) If $f(x, y) = ln(y - x)$, then find domain of it.

(b) Define level curve of a function $f(x, y)$ at a constant C.

(c) Find f_x if $f(x, y) = (\sin x^2)\cos y$.

Contd.

(d) If $f(x, y) = \sin xy$, then df is

(i) $y\cos xy\,dx + x\cos xy\,dy$

(ii) $y\cos xy\,dy + x\cos xy\,dx$

(iii) $y\cos x\,dx + x\cos y\,dy$

(iv) $\cos xy\,dx + \cos xy\,dy$

(Choose the correct option)

(e) Evaluate $\frac{\partial(x, y)}{\partial(r, \theta)}$ for the transformation $x = r\cos\theta,\ y = r\sin\theta$.

(f) If $P_0(x_0, y_0)$ is a critical point of $f(x, y)$ and f has continuous 2nd order partial derivatives in a disk centered at (x_0, y_0) and $D = f_{xx}f_{yy} - f_{xy}^2$, then a relative minimum occurs at P_0, if

(i) $D(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) < 0$

(ii) $D(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) > 0$

(iii) $D(x_0, y_0) < 0$ and $f_{yy}(x_0, y_0) < 0$

(iv) $D(x_0, y_0) < 0$ and $f_{yy}(x_0, y_0) > 0$

(Choose the correct option)

(g) The curl of a vector field

$$V(x, y, z) = u(x, y, z)i + v(x, y, z)j + w(x, y, z)k$$ is

(i) $\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)i + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)j + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)k$

(ii) $\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)i + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)j + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)k$

(iii) $\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right)i + \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)j + \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)k$

(iv) $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)i + \left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right)j + \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)k$

(Choose the correct option)

(h) Define work as a line integral.

(i) State Green's theorem on a simply connected region D.

(j) If a vector field F and curl F are both continuous in a simply connected region D on $\mathbb{R}^3$, then F is conservative in D if curl $F \neq 0$. State whether this statement is true *or* false.

2. Answer the following questions : 2×5=10

(a) Show that the function f is continuous at (0, 0) where

$$f(x, y) = \begin{cases} y \sin \frac{1}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

(b) Compute the slope of the tangent line to the graph of $f(x, y) = x^2 \sin(x + y)$ at the point $P\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$ in the direction parallel to the XZ plane.

(c) Evaluate $\iint\limits_R \frac{2xy}{x^2+1} dA$ where $0 \leq x \leq 1$, $1 \leq y \leq 3$.

(d) Show that $div\ F = 0$ and $curl\ F = 0$, if F is a constant vector field.

(e) Evaluate $\int_0^2 \int_0^1 \int_{-1}^2 8x^2yz^3 dx\, dy\, dz$.

3. Answer **any four** questions : 5×4=20

(a) (i) Find $\frac{\partial w}{\partial r}$ where $w = e^{2x-y+3z^2}$ and $x = r + s - t$, $y = 2r - 3s$, $z = \cos rst$. 3

(ii) Show that $\lim\limits_{(x, y) \to (0, 0)} \frac{x+y}{x-y}$ does not exist. 2

(b) (i) If f has a relative extremum at $P_0(x_0, y_0)$ and both f_x and f_y exist at (x_0, y_0), then prove that $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. 2

(ii) Discuss the nature of the critical points of the function $f(x, y) = (x-2)^2 + (y-3)^4$. 3

(c) Use a polar double integral to show that a sphere of radius a has volume $\frac{4}{3}\pi a^3$.

(d) An object moves in the force field $F = y^2 i + 2(x+1)yj$. How much work is performed as the object moves from the point (2, 0) counterclockwise along the elliptical path $x^2 + 4y^2 = 4$ to (0, 1), then back to (2, 0) along the line segment joining the two points.

(e) (i) Evaluate $\int_C (x^2 + y^2)dx + 2xy\,dy$ where C is the quarter circle $x^2 + y^2 = 1$ from (1, 0) to (0, 1). 3

(ii) A wire has the shape of the curve $x = \sqrt{2}\sin t,\ y = \cos t,\ z = \cos t$ for $0 \le t \le \pi$. If the wire has density $\delta(x, y, z) = xyz$ at each point (x, y, z), find its mass. 2

(f) Find the volume of the solid in the first octant that is bounded by the cylinder $x^2 + y^2 = 2y$, the half cone $z = \sqrt{x^2 + y^2}$ and the xy-plane.

4. Answer the following questions : 10×4=40

(a) (i) Let $f(x, y)$ be a function that is differentiable at $P_0(x_0, y_0)$. Prove that f has a directional derivative in the direction of the unit vector $u = u_1 i + u_2 j$ given by

$$D_u f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

3

(ii) Find the directional derivative of $f(x, y) = ln(x^2 + y^2)$ at $P_0(1, -3)$ in the direction of $v = 2i - 3j$ using the gradient formula. 3

(iii) Find the equations of the tangent plane and the normal line to the cone $z^2 = x^2 + y^2$ at the point where $x = 3,\ y = 4$ and $z > 0$. 4

OR

(i) Prove that if $f(x, y)$ is differentiable at (x_0, y_0), then it is continuous there. 4

Contd.

(ii) When two resistances R_1 and R_2 are connected in parallel, the total resistance R satisfies $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$.

If R_1 is measured as 300 ohms with a maximum error of 2% and R_2 is measured as 500 ohms with a maximum error of 3%, then find the maximum percentage error in R. 6

(b) (i) Use the method of Lagrange multipliers to minimize $f(x, y) = x^2 - xy + 2y^2$ subject to $2x + y = 22$. 5

(ii) Find all relative extrema and saddle points on the graph of $f(x, y) = x^2y^4$. 5

OR

(i) Find the absolute extrema of the function $f(x, y) = e^{x^2-y^2}$ over the disk $x^2 + y^2 \le 1$. 6

(ii) Suppose E be an extreme value of f subject to the constraint $g(x, y) = C$. Prove that the Lagrange multiplier λ is the rate of change of E with respect to C. 4

(c) (i) Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2}\, dy\, dx$ by converting to polar coordinates. 5

(ii) Evaluate $\iiint_D e^z\, dv$ where D is the region described by the inequalities $0 \le x \le 1$, $0 \le y \le x$ and $0 \le x \le x + y$. 5

OR

(i) Find the volume of the solid bounded above by the plane $z = y$ and below in the xy-plane by the part of the disk $x^2 + y^2 \le 1$ in the 1st quadrant. 5

Contd.

(ii) Evaluate : $\iiint_D x\,dV$ where D is the solid in the 1st octant bounded by the cylinder $x^2 + y^2 = 4$ and the plane $2y + z = 4$. 5

(d) (i) Let C be a piecewise smooth curve that is parameterized by a vector function $R(t)$ for $a \le t \le b$ and let F be a vector field that is continuous on C. If f is a scalar function such that $F = \nabla f$, then prove that $\int_C F.dR = f(Q) - f(P)$

where $Q = R(b)$ and $P = R(a)$ are the end points of C.

Using it evaluate the line integral $\int_C F.dR$, where

$F = \nabla\left(e^x \sin y - xy - 2y\right)$ and C is the path described by

$$R(t) = \left[t^3 \sin\frac{\pi}{2}t\right]i - \left[\frac{\pi}{2}\cos\left(\frac{\pi}{2}t + \frac{\pi}{2}\right)\right]j$$

for $0 \le t \le 1$ 5+3=8

(ii) Determine whether the vector field $F(x, y) = \dfrac{(y+1)i - xj}{(y+1)^2}$ is conservative. 2

OR

(i) Evaluate $\oint_C \left(\frac{1}{2}y^2dx + zdy + xdz\right)$ where C is the curve of intersection of the plane $x + z = 1$ and the ellipsoid $x^2 + 2y^2 + z^2 = 1$, oriented counterclockwise as viewed from above. 6

(ii) Evaluate $\iint_S F.NdS$ where

$F = x^2i + xyj + x^3y^3k$ and S is the surface of the tetrahedron bounded by the plane $x + y + z = 1$ and the coordinate planes with outward unit normal vector N. 4

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3 (Sem–4/CBCS) MAT HC 2

2024

MATHEMATICS

(Honours Core)

Paper : MAT–HC–4026

(Numerical Methods)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following as directed : 1×7=7

   (a) Name the three basic components of an algorithm.

   (b) Show $\nabla E \equiv \Delta$.

   (c) Write down the Lagrangian linear interpolation formula at the points x_0 and x_1 with corresponding function values f_0 and f_1.

Contd.

(d) What is the order of convergence of secant method?

(e) The approximation formula for finding the derivative at x_0 given by

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2} f''(\xi), \quad x_0 < \xi < x_{0+h}$$

is a

(i) backward difference approximation formula of first order of approximation

(ii) forward difference approximation formula of second order of approximation

(iii) forward difference approximation formula of first order of approximation

(iv) None of the above

(Choose the correct option)

(f) What is numerical integration? What is its general form?

(g) Name a method for approximating a solution to an initial value problem.

2. Answer the following questions : 2×4=8

(a) Compute the following limit and determine the rate of convergence

$$\lim_{x \to 0} \frac{e^x - 1}{x}.$$

(b) Prove $(I + \Delta)(I - \nabla) \equiv I$.

(c) Show that LU decomposition of a matrix is unique up to scaling by a diagonal matrix.

(d) Find the approximate value of $\int_0^1 \frac{dx}{1+x}$ by Simpson's rule.

3. Answer ***any three*** : 5×3=15

(a) Construct an iteration function corresponding to the given function $f(x) = x^3 - x^2 - 10x + 7$.

Use the fixed point iteration scheme with initial approximation as $P_0 = 1$ and perform three iterations to approximate the root of $f(x) = 0$.

(b) Using the data given below form the divided difference table and use it to construct the Newton form of the interpolating polynomial :

x	-1	0	1	2
y	5	1	1	11

(c) Use four iterations of Newton's method to approximate the root of the equation

$$f(x) = x^3 + 2x^2 - 3x - 1$$

in the interval (1, 2) starting with an initial approximation of $P_0 = 1$.

(d) Derive the second order central difference approximation for first derivative including error term given by

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0 - h)}{2h} - \frac{h^2}{6} f'''(\xi)$$

(e) (i) Name the measures by which errors are quantified. Write down the expressions for the same.

(ii) Prove that $\Delta^n f(x_i) = (E - I)^n f(x_i)$

4. Answer ***any three*** : 10×3=30

(a) What is Theoretical Error Bound ? Show that the Bisection Method for approximating a root of the equation $f(x) = 0$ always converges. Find the order of convergence of the Bisection Method. 1+6+3=10

(b) Verify that the equation $x^3 + x^2 - 3x - 3 = 0$ has a root in the interval (1, 2). Given that the exact root is $x = \sqrt{3}$, perform the first three iterations of the Regula-Falsi method. What is the computable estimate for $|e_n|$, the error obtained in nth step by this method. Verify that the absolute error in the third approximation satisfies the error estimate. 1+6+3=10

(c) What is an interpolating polynomial? Determine the interpolation error when a function is approximated by a constant polynomial. Mention an advantage and a disadvantage of Lagrangian form of the interpolating polynomial. Derive the Lagrangian interpolating polynomial for the given data : 1+2+2+5=10

x	-2	-1	0	1	2	3
y	39	3	-1	-3	-9	-1

(d) What are two different classes of methods for solving a linear system of equations. Name one method of each type. What do you mean by an LU decomposition of square matrix A.

Solve the following system using LU decomposition : 1+1+8=10

$$2x_1 + 7x_2 + 5x_3 = -4$$
$$6x_1 + 20x_2 + 10x_3 = -16$$
$$4x_1 + 3x_2 = -7$$

(e) (i) Derive the basic Trapezoidal rule for integrating $\int_a^b f(x)dx$. 6

(ii) Use appropriate first order approximation formulas to find derivatives of the values of $f(x)$ at the points $x = 0{\cdot}5$, $x = 0{\cdot}6$ and $x = 0{\cdot}7$. 4

x	$f(x)$	$f'(x)$
0·5	0·4794	?
0·6	0·5646	?
0·7	0·6442	?

(f) What is the basic problem that is solved by Euler's method? Derive Euler's method. Given that the exact solution to $\frac{dx}{dt} = \frac{t}{x}$ is $x(t) = \sqrt{t^2 + 1}$, find the absolute error at each step that is obtained by solving

$$\frac{dx}{dt} = \frac{t}{x},\ 0 \le t \le 1{\cdot}0,\ x(0) = 1,\ h = 0{\cdot}5$$

by Euler's method. 1+4+5=10

Total number of printed pages–8

3 (Sem–4/CBCS) MAT HC 3

2024

MATHEMATICS

(Honours Core)

Paper : MAT–HC–4036

(Ring Theory)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : $1\times10=10$

   (a) Give an example to show that for two non-zero elements a and b of a ring R, the equation $ax = b$ can have more than one solution.

   (b) How many nilpotent elements have in an integral domain ?

Contd.

(c) Which of the following statements is not true?

(i) $\langle 5 \rangle$ is a prime ideal of Z.

(ii) $\langle 5 \rangle$ is a maximal ideal of Z.

(iii) $\langle 5 \rangle$ is a maximal ideal of Z_{20}.

(iv) $\dfrac{Z}{5Z}$ is an integral domain.

(d) Define prime ideal of a ring.

(e) Give example of a commutative ring without zero divisor that is not an integral domain.

(f) Consider the polynomial

$f(x) = 4x^3 + 2x^2 + x + 4$ and

$g(x) = 3x^4 + 3x^3 + 3x^2 + x + 4$ in Z_5.

Compute $f(x) + g(x)$.

(g) Write $f(x) = x^3 + x^2 + x + 1 \in Z_2[x]$ as a product of irreducible polynomial over Z_2.

(h) Which of the following is a primitive polynomial?

(i) $2x^3 + 4x^2 + 6x + 10$

(ii) $5x^2 - 30x - 20$

(iii) $2x^4 + 3x^3 + 5x^2 - 7x + 11$

(iv) $3x^2 - 3x + 3$

(i) State whether the following statement is true or false :

"A polynomial $f(x)$ in $Z[x]$ which is reducible over Z is also reducible over Q."

(j) Choose the correct statement :

(i) Every Euclidean domain is a unique factorization domain.

(ii) Every principal ideal domain is a Euclidean domain.

(iii) Every unique factorization domain is a Euclidean domain.

(iv) Every unique factorization domain is a principal ideal domain.

2. Answer the following questions : 2×5=10

(a) If a and b are *two* idempotents in a commutative ring, then show that $a+b-ab$ is also an idempotent element.

(b) Show that every non-zero element of Z_n is a unit or a zero divisor.

(c) Show that every ring homomorphism $f: Z_n \to Z_n$ is of the form $f(x)=ax$ where $a=a^2$.

(d) Find the zeros of $f(x)=x^2+3x+2$ in Z_6.

(e) Let D be an integral domain and $a,b \in D$. If $\langle a \rangle = \langle b \rangle$, then show that a and b are associates.

3. Answer ***any four*** questions : 5×4=20

(a) The operations $\oplus$ and $\otimes$ defined on the set Z of integers by $a \oplus b = a+b-1$ and $a \otimes b = a+b-ab$. Show that $(Z, \oplus, \otimes,)$ is a ring with unity.

(b) Find all ring homomorphism from $Z \oplus Z$ to Z.

(c) Let R be a commutative ring with unity. Show that an ideal A of R is prime if and only if the quotient ring $\frac{R}{A}$ is an integral domain.

(d) Define principal ideal domain. Show that if F is a field, then $F[x]$ is a principal ideal domain. 1+4=5

(e) Show that every Euclidean domain is a principal ideal domain.

(f) Show that the number of reducible polynomials over Z_p of the form x^2+ax+b is $\frac{p(p+1)}{2}$.

4. Answer the following questions : 10×4=40

(a) (i) Let R be a commutative ring with unity. Show that the set

$$R[x] = \{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \,/\, a_i \in R,$$

n is a non-negative integer$\}$ is a ring. Also show that if R is an integral domain, then $R[x]$ is also an integral domain. 5+2=7 *(ii)* Let $f(x)=5x^4+3x^3+1$ and $g(x)=3x^2+2x+1$ in $Z_7[x]$.

Determine the quotient and remainder upon dividing $f(x)$ by $g(x)$. 3

Or

(i) Show that

$Z\left|\sqrt{2}\right|=\left\{a+b\sqrt{2} : a,b\in Z\right\}$ and

$$H=\left\{\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} / a,b\in Z\right\}$$

are isomorphic as ring. 4

(ii) If a, b be *any two* ring elements and m and n be *any two* integers, then show that $(m.a)(n.b)=(mn).(ab)$ 6

(b) *(i)* Define maximal ideal of a ring. Let A be an ideal of a commutative ring with unity R. Prove that $\frac{R}{A}$ is a field if and only if A is maximal. 1+6=7

(ii) Let R be a commutative ring and A be any subset of R. Show that the nil-radical of A,

$$N(A)=\left\{r\in R / r^n\in A \text{ for some } n\in N\right\}$$

is an ideal of R. 3

Or

(i) Let φ be a ring homomorphism from R to S. Then the mapping from $\frac{R}{\ker(\varphi)}$ to $\varphi(R)$, given by $r+\ker(\varphi)\to\varphi(r)$ is an isomorphism, i.e., $\frac{R}{\ker(\varphi)}\approx\varphi(R)$. 6

(ii) Let ϕ be a ring homomorphism from a ring R to a ring S. Let B be an ideal of S. Then $\phi^{-1}[B]=\{r\in R : \phi(r)\in B\}$ is an ideal of R. 4

(c) If F is a field and $p(x)\in F[x]$, then prove that $\frac{F[x]}{\langle p(x)\rangle}$ is a field if and only if $p(x)$ is irreducible over F.

Or

Let F be a field. If $f(x) \in F[x]$ and degree $f(x)$ is 2 or 3, then $f(x)$ is reducible over F if and only if $f(x)$ has a zero in F. Is the result true when degree $f(x)$ is greater then 3 ? Justify. 7+3=10

(d) In a principal ideal domain, show that an element is irreducible if and only if it is prime. Use this result to show that $Z[\sqrt{-3}] = \{a + b\sqrt{-3} : a, b \in Z\}$ is not a principal ideal domain. 7+3=10

Or

(i) In a principal ideal domain, show that any strictly increasing chain of ideals $I_1 \subset I_2 \subset \ldots$ must be finite in length. 5

(ii) Let ϕ be a onto ring homomorphism from a ring R to a ring S. Then prove that ϕ is an isomorphism if and only if $\ker(\phi) = \{0\}$. 3

(iii) Determine all ring homomorphism from the ring of integers Z to itself. 2
