Total number of printed pages–16

3 (Sem–6/CBCS) MAT HC 1 (N/O)

2024

MATHEMATICS

(Honours Core)

Paper : MAT–HC–6016

New Syllabus

(Riemann Integration and Metric Spaces)

Full Marks : 80

Time : Three hours

Old Syllabus

(Complex Analysis)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

Contd.

New Syllabus

(Riemann Integration and Metric Spaces)

Full Marks : 80

Time : Three hours

1. Answer the following as directed : 1×10=10

(a) Let $f:[a,b]\to R$ be a bounded function and P, Q are partitions of $[a,b]$. If Q is a refinement of P, then

(i) $L(f,Q)\le L(f,P)$

(ii) $U(f,P)\le U(f,Q)$

(iii) $U(f)\le L(f)$

(iv) $L(f)\le U(f)$

(Choose the correct option)

(b) Find the value of $\int_0^\infty e^{-x}dx$.

(c) Show that $\Gamma(1)=1$.

(d) Define Cauchy sequence in a metric space.

(e) State whether the following statement is true **or** false :

"Each subset of a discrete metric space is open."

(f) If the mapping $d:R^2\times R^2\to R$ is defined as $d((x_1,y_1),(x_2,y_2))=|x_1-x_2|$, then which one of the following statements is true ?

(i) d is the usual metric on R^2

(ii) d is uniform metric on R^2

(iii) d is a pseudo metric on R^2

(iv) None of the above statements is true

(g) Which of the following statements is not true ?

(i) In a metric space countable union of open sets is open

(ii) In a metric space finite union of closed sets is closed

(iii) A non-empty subset of a metric space is closed if and only if its complement is open

(iv) None of the above statements is true

(h) When is a metric space said to be connected.

(i) State whether the following statement is true **or** false :

"Image of an open set under a continuous function is open."

(j) Under what condition the metric spaces (X, d_X) and (Y, d_Y) are said to be equivalent ?

2. Answer the following questions : 2×5=10

(a) Let $u, v : [a, b] \to R$ be differentiable and u', v' are integrable on $[a, b]$. Then show that

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx.$$

(b) Show that a subset F of a metric space (X, d) is closed if and only if $\overline{F} = F$.

(c) Let (Y, d_Y) be a subspace of a metric space (X, d_X) and $S_X(z, r)$ and $S_Y(z, r)$ are open balls with center at $z \in Y$ and radius r in the metric space (X, d_X) and (Y, d_Y) respectively.

Prove that $S_Y(z, r) = S_X(z, r) \cap Y$.

(d) Show that the image of a Cauchy sequence under uniformly continuous function is again a Cauchy sequence.

(e) Show that a contraction mapping on a metric space is uniformly continuous.

3. Answer ***any four*** questions : 5×4=20

(a) Show that a bounded function $f : [a, b] \to R$ is integrable if and only if for each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$.

(b) Let g be a continuous function on the closed interval $[a, b]$ and the function f be continuously differentiable on $[a, b]$. Further if f' does not change sign on $[a, b]$, then show that there exists $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx.$$

Contd.

(c) Let (X, d) be a metric space and the function $d^*: X \times X \to R$ is defined as

$$d^*(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \forall x, y \in X$$

Show that (X, d^*) is a bounded metric space.

(d) Let Y be a non-empty subset of the metric space (X, d). Prove that the subspace (Y, d_Y) is complete if and only if Y is closed on (X, d).

(e) Show that composition of two uniformly continuous functions is also uniformly continuous.

(f) Show that a metric space (X, d) is disconnected if and only if there exists a continuous function of (X, d) onto the discrete two element space (X_0, d_0), i.e., $X_0 = \{0, 1\}$ and d_0 is the discrete metric on X_0.

4. Answer the following questions : 10×4=40

(a) Let f be a function on an interval J with nth derivative $f^{(n)}$ continuous on J. If $a, b \in J$, then show that

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \cdots + \frac{f^{(n)}(a)}{(n-1)!}(b-a)^{n-1} + R_n$$

$$\text{where, } R_n = \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt$$

Or

Let $f:[0, 1] \to R$ be continuous and $c_i \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$, $n \in N$.

Then show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(c_i) = \int_0^1 f(x)\,dx.$$

Hence show that

$$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{r}{r^2 + n^2} = \log\sqrt{2}.$$ 5+5=10

(b) Let l_p $(p \geq 1)$ be the set of all sequences of real numbers such that if $x = \{x_n\}_{n\geq 1} \in l_p$, then $\sum_{i=1}^{\infty} |x_i|^p < \infty$.

Prove that the function $d : l_p \times l_p \to R$ defined by $d(x, y) = \left\{ \sum_{n=1}^{\infty} |x_n - y_n|^p \right\}^{\frac{1}{p}}$

is a metric on l_p. Also show that l_p is a complete metric space. 4+6=10

Or

(i) Let (X, d) be a metric space and $\{x_n\}_{n\geq !}$, $\{y_n\}_{n\geq !}$ be two sequences in X such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Then show that $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$. 4

(ii) Let (X, d) be a metric space and Y a subspace of X. Let Z be a subset of Y. Then show that Z is closed in Y if and only if there exists a closed set $F \subseteq X$ such that $Z = F \cap Y$. 6

(c) What are meant by contraction mapping and fixed point of a contraction mapping in a metric space? If $T : X \to X$ is a contraction mapping on a complete metric space, then show that T has a unique fixed point. (1+1)+8=10

Or

If (X, d) be a metric space, then show that the following statements are equivalent:

(i) (X, d) is disconnected.

(ii) There exist two non-empty disjoint subsets A and B, both open in X, such that $X = A \cup B$.

(iii) There exist two non-empty disjoint subsets A and B, both closed in X, such that $X = A \cup B$.

(iv) There exists a proper subset of X that is both open and closed in X.

(d) (i) Let $f : [a, b] \to R$ be integrable and $F(x) = \int_a^x f(t)\,dt$; $x \in [a, b]$. Show that F is continuous on $[a, b]$. Also show that F is differentiable at $x \in [a, b]$ if f is continuous at $x \in [a, b]$ and $F'(x) = f(x)$. 7

(ii) Let (X, d) be a metric space and $\rho : X \times X \to R$ be defined by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)} \quad ; \quad x, y \in X .$$

Show that d and ρ are equivalent metrics. 3

Or

(iii) Show that a subset G of a metric space (X, d) is open if and only if it is the union of all open balls contained in G. 5

(iv) Give example, with justification, of a homeomorphism from a metric space onto another metric space which is not an isometry. 5

Old Syllabus

Full Marks : 60

(Complex Analysis)

Time : Three hours

1. Answer the following as directed : 1×7=7

(a) Determine the accumulation point of the set $z_n = \frac{i}{n} \, (n = 1, 2, 3, \cdots)$

(b) Describe the domain of $f(z) = \frac{z}{z + \bar{z}}$.

(c) Define an entire function.

(d) Determine the singular points of

$$f(z) = \frac{2z + 1}{z\left(z^2 + 1\right)}$$

(e) The value of loge is

(i) 1

(ii) $1 + 2n\pi i$

(iii) $2n\pi i$

(iv) 0

(Choose the correct option)

(f) $\lim_{n\to\infty}\left(-2+i\,\frac{(-1)^n}{n^2}\right)$ is equal to

(i) 0 (ii) -2

(iii) $-2+i$ (iv) limit does not exist

(Choose the correct option)

(g) The power expression for $\cos z$ is

(i) $\frac{e^z+e^{-z}}{2}$ (ii) $\frac{e^{iz}+e^{-iz}}{2}$

(iii) $\frac{e^{iz}+e^{-iz}}{2i}$ (iv) $\frac{e^z-e^{-z}}{2}$

(Choose the correct option)

2. Answer the following questions : 2×4=8

(a) Sketch the set

$$|z-1+i|\le 1$$

(b) Prove that $f'(z)$ exists every where for the function $f(z)=iz+2$.

(c) If $f(z)=\frac{z}{\bar{z}}$, prove that $\lim_{z\to 0} f(z)$ does not exist.

(d) Evaluate $\int_1^2\left(\frac{1}{t}-i\right)^2 dt$.

3. Answer **any three** questions from the following : 5×3=15

(a) (i) Show that if e^z is real, then $Im\,z=n\pi\,(n=0,\ \pm1,\ \pm2,\cdots)$ 3

(ii) Show that $\exp(2\pm 3\pi i)=-e^2$. 2

(b) Suppose that $f(z)=u(x,y)+iv(x,y)$, where $z=x+iy$ and $z_0=x_0+iy_0$, $w_0=u_0+iv_0$. Then prove that

$$\lim_{z\to z_0} f(z)=w_0 \text{ if}$$

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y)=u_0 \text{ and}$$

$$\lim_{(x,y)\to(x_0,y_0)} v(x,y)=v_0$$

(c) Show that $f'(z)$ exists everywhere, when $f(z)=e^z$.

(d) Evaluate $\int_C \frac{dz}{z}$, where C is the top half of the circle $|z|=1$ from $z=1$ to $z=-1$.

Contd.

(e) Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Applying Cauchy's integral formula, evaluate $\int_C \frac{e^{-z}dz}{z - (\pi i/2)}$.

4. Answer **either** (a) **or** (b) and (c):

(a) Suppose that $f(z) = u(x, y) + iv(x, y)$ and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Prove that the first order partial derivatives of u and v must exist at (x_0, y_0) and they must satisfy the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ there.

Also show that $f'(z) = u_x + iv_x = v_y - iu_y$ where partial derivatives are to be evaluated at (x_0, y_0). 10

Or

(b) If z_0 and w_0 are points in the z-plane and w-plane respectively, then prove that $\lim_{z \to z_0} f(z) = \infty$ if and only if

$$\lim_{z \to z_0} \frac{1}{f(z)} = 0$$

Hence show that $\lim_{z \to -1} \frac{iz + 3}{z + 1} = \infty$

4+2=6

(c) If $w = f(z) = \bar{z}$, examine whether $\frac{dw}{dz}$ exists or not. 4

5. Answer **either** (a) **or** (b):

(a) Let C denote a contour of length L and suppose that a function $f(z)$ is piecewise continuous on C. If M is a non-negative constant such that $|f(z)| \leq M$ for all points z on C at which $f(z)$ is defined, then prove that

$$\left| \int_C f(z)dz \right| \leq ML$$

Hence show that

$\left| \int_C \frac{dz}{z^2 - 1} \right| \leq \pi/3$, where C is the arc of the semicircle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. 10

Or

(b) State and prove Liouville's theorem.

Contd.

6. Answer ***either*** *(a)*, *(b)*, *(c)* **or** *(d)* :

(a) Prove that if a series of complex numbers converges, then the nth term converges to zero as n tends to infinity. 3

(b) Test the convergency of the sequence

$$z_n = \frac{1}{n^3} + i\ (n = 1, 2, \cdots)$$ 3

(c) Find Maclaurin's series for the entire function $f(z) = l^z$. 4

Or

(d) Suppose that a function f is analytic throughout a disc $|z - z_0| < R_0$ centred at z_0 and with radius R_0. Then prove that $f(z)$ has a power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (|z - z_0| < R_0)$$

where $a_n = \dfrac{f^n(z_0)}{n!} (n = 0, 1, 2, \cdots)$

Total number of printed pages–7

3 (Sem–6/CBCS) MAT HC 2

2024

MATHEMATICS

(Honours Core)

Paper : MAT–HC–6026

(Partial Differential Equations)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following: 1×7=7

(i) Under which of the following conditions does arbitrary constant elimination usually produce more than one partial differential equation of order one ?

(a) The number of arbitrary constants is less than that of independent variables

Contd.

(b) The number of arbitrary constants equals the number of independent variables

(c) The number of arbitrary constants is more than that of independent variables

(d) Both (a) and (b)

(Choose the correct answer)

(ii) State True or False :

$\left(x^2 - yz\right)p + \left(y^2 - zx\right)q = z^2 - xy$ is a first order quasi-linear partial differential equation.

(iii) The order of $p \tan y + q \tan x = \sec^2 z$ is ______.

(iv) The Charpit's method is used for

(a) general solution

(b) complete solution

(c) singular solution

(d) complete integral

(Choose the correct answer)

(v) Jacobi's auxiliary equations for $p_1 x_1 + p_2 x_2 - p_3^2 = 0$ are ______.

(vi) What are the characteristic equations of $u_x - u_y = u$?

(vii) The equation $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0$ is

(a) parabolic for $x \neq 0$ and $y \neq 0$ only

(b) parabolic for $x = 0$ and $y = 0$ only

(c) parabolic everywhere

(d) parabolic nowhere

(Choose the correct answer)

2. Answer in short : 2×4=8

(i) Consider an equation of the form $a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$, where its coefficients a, b and c are functions of x, y and u. Is it linear ? Justify your answer.

(ii) Eliminate the arbitrary function f from $z = x^n f\left(\frac{y}{x}\right)$ to form a partial differential equation.

(iii) Mention when Jacobi's method is used. Name an advantage of Jacobi's method over Charpit's method.

(iv) Construct an example of a partial differential equation that is elliptic in one domain but hyperbolic in another.

3. Answer **any three** : 5×3=15

(i) Find the partial differential equation that all surfaces of revolution satisfy with the z-axis as the axis of symmetry, along with a suitable explanation.

(ii) Find the general solution of the differential equation

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x+y)z.$$

(iii) Find the integral surface of the equation

$$(x-y)y^2 p + (y-x)x^2 q = (x^2+y^2)z$$

through the curve $xz = a^3$, $y = 0$.

(iv) Reduce the equation

$$u_x + 2xyu_y = x$$

to canonical form, and obtain the general solution.

(v) Discuss the general solution of $Au_{xx} + Bu_{xy} + Cu_{yy} = 0$ with constant coefficients in hyperbolic case.

4. Answer the following : 10×3=30

(i) Discuss briefly the essential steps in Charpit's method for solving partial differential equations. Use this method to solve the equation $p = (z + qy)^2$.

Or

Show that the only integral surface of the equation $2q(z - px - qy) = 1 + q^2$ which is circumscribed about the paraboloid $2x = y^2 + z^2$ is the enveloping cylinder which touches it along its section by the plane $y + 1 = 0$.

(ii) Describe in brief the key components of the 'method of separation of variables'. Use this method suitably to solve the equation $u_x + u = u_y$, $u(x, 0) = 4e^{-3x}$.

Or

Use $v = \ln u$ and $v(x, y) = f(x) + g(y)$ to solve the equation $x^2u_x^2 + y^2u_y^2 = u^2$.

Also, discuss briefly the approach adopted to solve the above equation.

(iii) Consider the wave equation

$u_u - c^2u_{xx} = 0$, c is constant.

Establish that any general solution of this equation can be expressed as the sum of two waves, one travelling to the right with constant velocity c and the other travelling to the left with the same velocity c.

Or

Find the general solution of the following equations :

(a) $yu_{xx} + 3yu_{xy} + 3u_x = 0,\ y \neq 0$

(b) $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$

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