Total number of printed pages–16

3 (Sem–5/CBCS) MAT HC 1 (N/O)

2023

MATHEMATICS

(Honours Core)

OPTION–A

(For New Syllabus)

Paper : MAT–HC–5016

(Complex Analysis)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×7=7

(a) Which point on the Riemann sphere represents ∞ of the extended complex plane $\mathbb{C} \cup \{\infty\}$?

(b) A set $S \subseteq \mathbb{C}$ is closed if and only if S contains each of its ________ points. *(Fill in the gap)*

Contd.

(c) Write down the polar form of the Cauchy-Riemann equations.

(d) The function $f(z) = sinh z$ is a periodic function with a period ________ .

(Fill in the gap)

(e) Define a simple closed curve.

(f) Write down the value of the integral $\int_C f(z)\, dz$, where $f(z) = z e^{-2}$ and C is the circle $|z| = 1$.

(g) Find $\lim_{n\to\infty} z_n$, where $z_n = -1 + i\frac{(-1)^n}{n^2}$.

2. Answer the following questions : 2×4=8

(a) Let $f(z) = i\frac{z}{2}$, $|z| < 1$. Show that $\lim_{z\to 1} f(z) = \frac{i}{2}$, using $\varepsilon - \delta$ definition.

(b) Show that all the zeros of $sinh z$ in the complex plane lie on the imaginary axis.

(c) Evaluate the contour integral $\int_C \frac{dz}{z}$, where C is the semi circle $z = e^{i\theta}$, $0 \le \theta \le \pi$

(d) Using Cauchy's integral formula, evaluate

$\int_C \frac{e^{2z}}{z^4} dz$, where C is the circle $|z| = 1$.

3. Answer **any three** questions from the following : 5×3=15

(a) Find all the fourth roots of -16 and show that they lie at the vertices of a square inscribed in a circle centered at the origin.

(b) Suppose $f(z) = u(x, y) + iv(x, y)$, $(z = x + iy)$ and $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$. Then prove the following:

$\lim_{(x, y)\to(x_0, y_0)} u(x, y) = u_0$,

$\lim_{(x, y)\to(x_0, y_0)} v(x, y) = v_0$, if and only if $\lim_{z\to z_0} f(z) = w_0$.

(c) (i) Show that the function $f(z) = Re\, z$ is nowhere differentiable.

(ii) Let $T(z) = \frac{az+b}{cz+d}$, where $ad - bc \neq 0$.

Show that $\lim_{z \to \infty} T(z) = \infty$ if $c = 0$.

3+2=5

(d) Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. Show that

$$\left| \int_C \frac{z+4}{z^3 - 1} dz \right| \leq \frac{6\pi}{7}$$

(e) State and prove fundamental theorem of algebra.

4. Answer ***any three*** questions from the following : 10×3=30

(a) (i) Show that $exp(z + \pi i) = -exp(z)$ 1

(ii) Show that

$log(-1+i)^2 \neq 2 log(-1+i)$ 2

(iii) Show that

$|\sin z|^2 = \sin^2 x + \sinh^2 y$ 2

(iv) Show that a set $S \subseteq \mathbb{C}$ is unbounded if and only if every neighbourhood of the point at infinity contains at least one point of S. 5

(b) (i) Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$, $g'(z_0)$ exist with $g'(z_0) \neq 0$. Using the definition of derivative show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$ 5

(ii) Show that

$$z^2 e^{3z} = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n,$$

where $|z| < \infty$. 5

(c) State and prove Laurent's theorem.

(d) (i) Using definition of derivative, show that $f(z) = |z|^2$ is nowhere differentiable except at $z = 0$. 5

(ii) Define singular points of a function. Determine singular points of the functions :

$$f(z) = \frac{2z+1}{z(z^2+1)};$$

$$g(z) = \frac{z^3+i}{z^2-3z+2}$$ 1+4=5

(e) (i) Let $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D. Prove that the families of curves $u(x, y) = c_1$, $v(x, y) = c_2$ are orthogonal.

(ii) Let C denote a contour of length L and suppose that a function $f(z)$ is piecewise continuous on C. If M is a non-negative constant such that

$|f(z)| \le M$ for all z in C

then show that

$$\left| \int_C f(z)\, dz \right| \le ML.$$ 5+5=10

(f) (i) Prove that two non-zero complex numbers z_1 and z_2 have the same moduli if and only if $z_1 = c_1 c_2$, $z_2 = c_1 \bar{c}_2$, for some complex numbers c_1, c_2. 4

(ii) Show that mean value theorem of integral calculus of real analysis does not hold for complex valued functions $w(t)$. 3

(iii) State Cauchy-Goursat theorem. 1

(iv) Show that $\lim_{z \to \infty} \frac{z^2+1}{z-1} = \infty$. 2

Contd.

OPTION-B

(For Old Syllabus)

(Riemann Integration and Metric Spaces)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×10=10

   (a) Write the statement of the First Fundamental Theorem of Calculus.

   (b) Evaluate $\int_0^\infty e^{-x}\,dx$, if it exists.

   (c) Prove that $\Gamma(1)=1$.

   (d) Define a complete metric space.

   (e) Describe an open ball in the discrete metric space (X,d).

   (f) $(A\cup B)^0$ need not be $A^0 \cup B^0$ — Justify it where A and B are subsets of a metric space (X,d).

   (g) Find the derived sets of the intervals $(0,1)$ and $[0,1]$.

   (h) Let A and B be two subsets of a metric space (X,d). Which of the following is not correct?

   (i) $A\subseteq B \Rightarrow A'\subseteq B'$

   (ii) $(A\cap B)' \subseteq A'\cap B'$

   (iii) $A'\cap B' \subseteq (A\cap B)'$

   (iv) $(A\cup B)' = A'\cup B'$

   (i) The Euclidean metric on $\mathbb{R}^n$ is defined as

   (i) $d(x,y)=\left\{\sum_{i=1}^{n}(x_i-y_i)^2\right\}^{\frac{1}{2}}$

   (ii) $d(x,y)=\left\{\sum_{i=1}^{n}|x_i-y_i|^p\right\}^{\frac{1}{p}}$ where $p\geq 1$

   (iii) $d(x,y)=\max\limits_{1\leq i\leq n}|x_i-y_i|$

   (iv) $d(x,y)=\sup\limits_{1\leq i\leq n}|x_i-y_i|$

   where $x=(x_1,x_2,\cdots x_n)$
   $y=(y_1,y_2,\cdots y_n)$

   are any two points in $\mathbb{R}^n$.

   (Choose the correct answer)

(j) Let (X, d_X) and (Y, d_Y) be two metric spaces and $f: X \to Y$ be continuous on X. Then for any $B \subseteq Y$.

(i) $f^{-1}(\overline{B}) \subset \overline{f^{-1}(B)}$

(ii) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$

(iii) $\overline{f(B)} \subset f(\overline{B})$

(iv) $f(\overline{B}) \subset \overline{f(B)}$

(Choose the correct answer)

2. Answer the following questions : 2×5=10

(a) Let $f(x) = x$ on $[0, 1]$ and

$$P = \left\{ x_i = \frac{i}{4},\ i = 0, 1, \cdots, 4 \right\}$$

Find $L(f, P)$ and $U(f, P)$.

(b) Let $f: [0, a] \to \mathbb{R}$ be given by $f(x) = x^2$. Find

$$\int_0^a f(x)\, dx$$

(c) Let (X, d) be a metric space and A, B be subsets of X. Prove that $(A \cap B)^0 = A^0 \cap B^0$.

(d) If A is a subset of a metric space (X, d), prove that $d(A) = d(\overline{A})$.

(e) Let (X, d_X) and (Y, d_Y) be two metric spaces. Prove that if a mapping $f: X \to Y$ is continuous on X, then $f^{-1}(G)$ is open in X for all open subsets G of X.

3. Answer ***any four*** parts : 5×4=20

(a) Prove that $f(x) = x^2$ on $[0, 1]$ is integrable.

(b) Show that $\lim_{n \to \infty} \sum_{r=1}^{n} \frac{r}{r^2 + n^2} = \log \sqrt{2}$

(c) Let (X, d) be a metric space. Define $d': X \times X \to \mathbb{R}$ by

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)} \text{ for all}$$

$x, y \in X$. Prove that d' is a metric on X.

(d) Let $X = c[a, b]$ and

$d(f, g) = sup\{|f(x) - g(x)| : a \le x \le b\}$

be the associated metric where $f, g \in X$. Prove that (X, d) is a complete metric space.

(e) Let (X, d) be a metric space. Prove that a finite union of closed sets is closed.

Infinite union of closed sets need not to closed — Justify it. 3+2=5

(f) Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \to Y$ be uniformly continuous. If $\{x_n\}_{n \ge 1}$ is a Cauchy sequence in X, prove that $\{f(x_n)\}_{n \ge 1}$ is a Cauchy sequence in Y.

4. Answer ***any four*** parts : 10×4=40

(a) (i) Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that f is integrable. 5

(ii) Discuss the convergence of the integral $\int_1^{\infty} \frac{1}{x^p} dx$ for various values at p. 5

(b) (i) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Prove that there exists $c \in [a, b]$ such that

$$\frac{1}{b-a} \int_a^b f(x)\, dx = f(c)$$

Using it prove that for $-1 < a < 0$ and $n \in \mathbb{N}$,

$$S_n = \int_a^0 \frac{x^n}{1+x} dx \to 0 \text{ as } n \to \infty$$

3+2=5

(ii) Let $f : [a, b] \to \mathbb{R}$ be monotone. Prove that there exists $c \in [a, b]$ such that

$$\int_a^b f(x)\, dx = f(a)(c-a) + f(b)(b-c)$$

5

(c) (i) Prove that a convergent sequence in a metric space is a Cauchy sequence.

Show that in the discrete metric space every Cauchy sequence is convergent. 3+2=5

(ii) Define an open set in a metric space (X, d).

Prove that in any metric space (X, d), each open ball is an open set. 1+4=5

(d) (i) Let (X, d) be a metric space and F be a subset of X. Prove that F is closed in X if and only if F^c is open in X. 5

(ii) Let (X, d) be a metric space and Y a subspace of X. Let Z be a subset of Y. Prove that Z is open in Y if and only if there exists an open set $G \subseteq X$ such that $Z = G \cap Y$. 5

(e) (i) Let (X, d_X) and (Y, d_Y) be metric spaces and $A \subseteq X$. Prove that a function $f : A \to Y$ is continuous at $a \in A$ if and only if whenever a sequence $\{x_n\}$ in A converges to a, the sequence $\{f(x_n)\}$ converges to $f(a)$. 6

(ii) Prove that a mapping $f : X \to Y$ is continuous on X if and only if $f^{-1}(F)$ is closed in X for all closed subsets F of Y. 4

(f) (i) Show that the function $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is not uniformly continuous. 5

(ii) Let (X, d) be a metric space and let $x \in X$ and $A \subseteq X$ be non-empty. Prove that $x \in \bar{A}$ if and only if $d(x, A) = 0$. 5

(g) (i) Define a connected set in a metric space.
Prove that if Y is a connected set in a metric space (X, d), then any set Z such that $Y \subseteq Z \subseteq \overline{Y}$, is connected. 1+4=5

(ii) Let (X, d) be a metric space. Prove that the following statements are equivalent :

(a) (X, d) is disconnected

(b) there exists a continuous mapping of (X, d) onto the discrete two element space (X_0, d_0). 5

(h) Let $(\mathbb{R}, d)$ be the space of real numbers with the usual metric. Prove that a subset I of $\mathbb{R}$ is connected if and only if I is an interval.

Total number of printed pages–8

3 (Sem–5/CBCS) MAT HC 2

2023

MATHEMATICS

(Honours Core)

Paper : MAT–HC–5026

(Linear Algebra)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : $1\times10=10$

(a) Let $A=\begin{pmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{pmatrix}$ and $\vec{u}=\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$.

Check whether $\vec{u}$ is in null space of A.

(b) Define subspace of a vector space.

(c) Give reason why $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$.

Contd.

(d) State whether the following statement is true **or** false :

"If dimension of a vector space V is $p > 0$ and S is a linearly dependent subset of V, then S contains more than p elements."

(e) If $\vec{x}$ is an eigenvector of A corresponding to the eigenvalue λ then what is $A^3\vec{x}$?

(f) When two square matrices A and B are said to be similar?

(g) If $\vec{v} = (1 \ -2 \ 2 \ 4)$ then find $\|\vec{v}\|$.

(h) Find a unit vector in the direction of $\vec{u} = \begin{bmatrix} 8/3 \\ 2 \end{bmatrix}$.

(i) Under what condition two vectors $\vec{u}$ and $\vec{v}$ are orthogonal to each other?

(j) Define orthogonal complement of vectors.

2. Answer the following questions : 2×5=10

(a) Show that the set $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$ is not a subspace of $\mathbb{R}^2$.

(b) Let $\vec{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $\beta = \{b_1, b_2\}$. Find the coordinate vector $[x]_\beta$ of $\vec{x}$ relative to β.

(c) Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

(d) Let P_2 be the vector space of all polynomials of degree less than equal to 2. Consider the linear transformation $T : P_2 \to P_2$ defined by $T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$. Find the matrix representation $[T]_\beta$ of T with respect to the base $\beta = \{1, t, t^2\}$.

(e) Show that the matrix $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$ has orthogonal columns.

3. Answer ***any four*** questions : 5×4=20

(a) Let $S=\{v_1, v_2, \ldots, v_p\}$ be a set in the vector space V and $H = span(S)$. Now if one of the vector in S, say v_k, is linear combination of the other vectors in S, then show that S is linearly dependent and the subset of $S_1 = S - \{v_k\}$ still span H. 2+3=5

(b) Show that the set of all eigenvectors corresponding to the distinct eigenvalues of a $n \times n$ matrix A is linearly independent.

(c) Let W be a subspace of the vector space V and S is a linearly independent subset of W. Show that S can be extended, if necessary, to form a basis for W and $dim\, W \le dim\, V$.

(d) If $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

(e) If $\bar{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\bar{u} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$ then find the orthogonal projection of $\bar{y}$ onto $\bar{u}$ and write $\bar{y}$ as the sum of two orthogonal vectors, one in $span\{\bar{u}\}$ and the other orthogonal to $\bar{u}$.

(f) If $W = span\{x_1, x_2\}$ where $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$, find a orthogonal basis for W.

Answer ***either*** *(a)* ***or*** *(b)* from each of the following questions : 10×4=40

4. (a) Find a spanning set for the null space of the matrix :

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Is this spanning set linearly independent? 8+2=10

(b) (i) If a vector space V has a basis of n vectors, then show that every basis of V must consist of exactly n vectors. 4

(ii) Find a basis for column space of the following matrix : 6

$$B = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

5. (a) Define eigenvalue and eigenvector of a matrix. Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$. 2+8=10

(b) Let T be a linear operator on a finite dimensional vector space V and let W denote the *T-cyclic* subspace of V generated by a non-zero vector $v \in V$. If $dim(W) = k$ then show that

(i) $\{v, T(v), T^2(v), \dots, T^{k-1}(v)\}$ is a basis for W.

(ii) If

$$a_0 v + a_1 T(v) + \dots + a_{k-1} T^{k-1}(v) + T^k(v) = 0,$$

then the characteristics polynomial of T_w is

$$f(t) = (-1)^k \left(a_0 + a_1 t + \dots + a_{k-1} t^{k-1} + t^k\right).$$

6+4=10

6. (a) (i) Define orthogonal set of vectors. Let $S = \{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_p\}$ is an orthogonal set of non-zero vectors in $\mathbb{R}^n$, then show that S is linearly independent. 1+4=5

(ii) For any symmetric matrix show that any two eigenvectors from different eigenspaces are orthogonal. 5

(b) Define inner product space. Show that the following is an inner product in $\mathbb{R}^2$

$$\langle u, v \rangle = 4u_1 v_1 + 5u_2 v_2$$

Where $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$

Also, show that in any inner product space V,

$$|\langle u, v \rangle| \le \|u\| \cdot \|v\|, \ \forall u, v \in V.$$

2+4+4=10

7. *(a)* *(i)* Consider the bases $\beta = \{b_1, b_2\}$ and $\gamma = \{c_1, c_2\}$ for $\mathbb{R}^2$ where

$$b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad c_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$$

and $c_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, find the change of coordinate matrix from γ to β and from β to γ. 5

(ii) Compute A^{10} where

$$A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}.$$ 5

(b) State Cayley-Hamilton theorem for matrices. Verify the theorem for the matrix $M = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and hence find M^{-1}.