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3 (Sem-3/CBCS) MAT HC 1

2023

MATHEMATICS

(Honours Core)

Paper : MAT-HC-3016

(Theory of Real Functions)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×10=10

   (a) Is 0 a cluster point of (0,1) ?

   (b) " If the limit of a function f at a point C of its domain does not exist, then f diverges at C." *(Write True or False)*

   (c) Define $\lim\limits_{x \to c} f(x) = \infty$, where $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ and $C \in \mathbb{R}$ is a cluster point of A.

Contd.

(d) Write sequential criterion for continuity.

(e) What do you mean by an unbounded function on a set ?

(f) Let $A, B \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be continuous on A and let $g : B \to \mathbb{R}$ be continuous on B. Under what condition $g \circ f : A \to \mathbb{R}$ is continuous on A ?

(g) "If a function is continuous then it is uniformly continuous."

(Write True or False)

(h) If functions $f_1, f_2, \ldots, f_n$ are differentiable at c, write the expression $(f_1 . f_2 \ldots . f_n)'(c)$.

(i) The function $f(x) = x$ is defined on the interval $I = [0,1]$. Is 0 a relative maximum of f ?

(j) Define Taylor's polynomial for a function f at a point x_0, supposing f has an nth derivative at x_0.

2. Answer the following questions : 2×5=10

(a) Use $\varepsilon - \delta$ definition of limit to show that $\lim\limits_{x \to 0} \frac{x^2}{|x|} = 0$.

(b) Show that the absolute value function $f(x) = |x|$ is continuous at every point $c \in \mathbb{R}$.

(c) Give an example of a function $f : [0,1] \to \mathbb{R}$ that is discontinuous at every point of $[0,1]$, but $|f|$ is continuous on $[0,1]$.

(d) "Continuity at a point is not a sufficient condition for the derivative to exist at that point." Justify your answer.

(e) Show that $\lim\limits_{x \to 0+} \frac{\sin x}{\sqrt{x}} = 0$.

3. Answer ***any four*** parts : 5×4=20

(a) Prove that a number $c \in \mathbb{R}$ is a cluster point of a subset A of $\mathbb{R}$ if and only if there exists a sequence $\{a_n\}$ in A such that $\lim a_n = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.

(b) Show that (using $\varepsilon-\delta$ definition of limit)

$$\lim_{x \to 2} \frac{x^3-4}{x^2+1} = \frac{4}{5}$$

(c) Prove that if $I=[a,b]$ is a closed bounded interval and if $f: I \to \mathbb{R}$ is continuous on I then f is bounded on I.

(d) Show that if f and g are uniformly continuous on a subset A of $\mathbb{R}$ then $f+g$ is uniformly continuous on A.

(e) Suppose that f is continuous on a closed interval $I=[a,b]$ and that f has a derivative in the open interval (a,b) Then there exists *at least one* point c in (a,b) such that

$$f(b)-f(a)=f'(c)(b-a).$$

(f) Let $f: I \to \mathbb{R}$ be differentiable on the interval I. Then prove that f is increasing if and only if $f'(x) \geq 0$ for all $x \in I$.

4. Answer **any four** parts : 10×4=40

(a) Prove that a real valued function f is continuous at $c \in \mathbb{R}$ if and only if whenever every sequence $\{c_n\}$, converging to c, then corresponding sequence $\{f(c_n)\}$ converges to $f(c)$.

(b) (i) Show that every infinite bounded subset of $\mathbb{R}$ has *at least one* limit point. 5

(ii) Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A. If $a \leq f(x) \leq b$ for all $x \in A$, $x \neq c$ and if $\lim_{x \to c} f(x)$ exist then prove that

$$a \leq \lim_{x \to c} f \leq b.$$ 5

(c) (i) Let $I=[a,b]$ be a closed bounded interval. Let $f: I \to \mathbb{R}$ be such that f is continuous. Prove that f is uniformly continuous on $[a,b]$.

5

(ii) Show that the function $f(x)=\frac{1}{x^2}$ is uniformly continuous on $A=[1,\infty)$. 5

(d) Let $I=[a,b]$ be a closed bounded interval and let $f:I\to\mathbb{R}$ be continuous on I. Then f has an absolute maximum and an absolute minimum on I.

(e) (i) Let I be a closed bounded interval and let $f:I\to\mathbb{R}$ be continuous on I. Then the set $f(I)=\{f(x):x\in I\}$ is a closed bounded interval. 6

(ii) Let $A,B\subseteq\mathbb{R}$ and let $f:A\to\mathbb{R}$ and $g:B\to\mathbb{R}$ be functions such that $f(A)\subseteq B$. If f is continuous at a point $c\in A$ and g is continuous at $b=f(c)\in B$, then show that the composition $g\circ f:A\to\mathbb{R}$ is continuous at c. 4

(f) (i) Let $I=[a,b]$ and let $f:I\to\mathbb{R}$ be continuous on I. If $f(a)<0<f(b)$ or if $f(a)>0>f(b)$, then prove that there exists a number $c\in(a,b)$ such that $f(c)=0$. 6

(ii) Use the definition to find the derivative of the function $f(x)=\frac{1}{\sqrt{x}}$ for $x>0$. 4

(g) (i) State and prove Taylor's theorem. 2+5=7

(ii) Using the Mean Value theorem prove that $|\sin x-\sin y|\le|x-y|$ for all x, y in $\mathbb{R}$. 3

(h) (i) Show that

$$1-\frac{1}{2}x^2\le\cos x$$

for all $x\in\mathbb{R}$ 5

(ii) Evaluate $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$ 5

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3 (Sem–3/CBCS) MAT HC 2

2023

MATHEMATICS

(Honours Core)

Paper : MAT–HC–3026

(Group Theory–1)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : $1 \times 10 = 10$

   (a) Define order of an element of a group.

   (b) In the group Q^* of all non-zero rational numbers under multiplication, list the elements of $\left\langle \frac{1}{2} \right\rangle$.

   (c) Find elements A, B, C in D_4 such that $AB = BC$ but $A \neq C$.

Contd.

(d) Define simple group.

(e) State Cauchy's theorem on finite Abelian group.

(f) State whether the following statement is true **or** false:
"If H is a subgroup of the group G and $a \in G$, then $Ha = \{ha : a \in G\}$ is also a subgroup of G."

(g) Write the order of the alternating group A_n of degree n.

(h) Give an example of an onto group homomorphism which is not an isomorphism.

(i) State whether the following statement is true **or** false :
"If the homomorphic image of a group is Abelian then the group itself is Abelian."

(j) Which of the following statement is true ?

(a) A homomorphism from a group to itself is called monomorphism.

(b) A one-to-one homomorphism is called epimorphism.

(c) An onto homomorphism is called endomorphism.

(d) None of the above

2. Answer the following questions : 2×5=10

(a) In D_3, find all elements X such that $X^3 = X$.

(b) Consider the group Z_2 under $+_2$ and Z_3 under $+_3$. List the elements of $Z_2 \oplus Z_3$ and find $|Z_2 \oplus Z_3|$.

(c) Express $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 1 & 4 & 3 & 2 \end{pmatrix}$ as product of transposition and find its order.

(d) If $\psi : G \to G'$ is a group homomorphism and e and e' be the identity elements of the group G and G' respectively then show that $\psi(e) = e'$.

(e) Show that in a group G, if the map $f : G \to G'$ defined by $f(x) = x^{-1}$, $\forall x \in G$ is a homomorphism then G is Abelian.

3. Answer ***any four*** questions : 5×4=20

(a) Let G be a group and H be a non-empty finite subset of G. Prove that H is a subgroup of G if and only if H is closed under the operation in G.

(b) If a is an element of order n in a group and k is a positive integer then prove that

$\langle a^k \rangle = \langle a^{gcd(n,k)} \rangle$ and

$|a^k| = \dfrac{n}{gcd(n,k)}$.

(c) Show that a subgroup H of a group G is a normal subgroup of G if and only if product of two right cosets of H in G is again a right coset of H in G.

(d) If a, n are two integers such that $n \geq 1$ and $gcd(a, n) = 1$, then prove that $a^{\phi(n)} \equiv 1 (mod\, n)$, where $\phi(n)$ is the Euler's phi function.

(e) Show that any finite cyclic group of order n is isomorphic to $\dfrac{\mathbb{Z}}{\langle n \rangle}$, where $\mathbb{Z}$ is the additive group of integers and $\langle n \rangle = \{0, n, 2n, \ldots\}$.

(f) Let $\sigma : G \to \overline{G}$ be a group homomorphism and $a, b \in G$.

(i) Show that $\sigma(a) = \sigma(b) \Leftrightarrow a \ker \sigma = b \ker \sigma$.

(ii) If $\sigma(g) = g'$ then show that $\sigma(g') = \{x \in G : \sigma(x) = g'\} = g \ker \sigma$.

2+3=5

Answer ***either*** (a) ***or*** (b) from the following questions : 10×4=40

4. (a) Describe the elements of D_4, the symmetries of a square. Write down a complete Cayley's table for D_4. Show that D_4 forms a group under composition of functions. Is D_4 an Abelian group? 2+3+4+1=10

(b) Prove that every subgroup of a cyclic group is cyclic. Also show that if $|\langle a \rangle| = n$, then the order of any subgroup of $\langle a \rangle$ is a divisor of n. Moreover, show that the group $\langle a \rangle$ has exactly one subgroup $\langle a^{\frac{n}{k}} \rangle$ of order k. Find the subgroup of Z_{30} which is of order 3. 4+2+3+1=10

5\. *(a)* Show that every quotient group of a cyclic group is cyclic. Give example to show that converse of this statement is not true in general. Find $\frac{\mathbb{Z}}{N}$ where $\mathbb{Z}$ is the additive group of integers and $N = \{5n : n \in \mathbb{Z}\}$. 4+3+3=10.

(b) *(i)* Show that every finite group can be represented as a permutation group. 7

(ii) Let $\phi : G \to \overline{G}$ be a group homomorphism and H be a subgroup of G. If $\overline{K}$ is a normal subgroup of $\overline{G}$ then show that $\phi^{-1}[\overline{K}] = \{k \in G : \phi(k) \in \overline{K}\}$ is a normal subgroup of G. 3

6\. *(a)* *(i)* State and prove Lagrange's theorem for the order of subgroup of a finite group. Is the converse true? Justify your answer. 1+5+1=7

(ii) List the elements of $\frac{\mathbb{Z}}{4\mathbb{Z}}$ and construct a Cayley's table for it. 3

(b) *(i)* Show that any two disjoint cycles commute. 5

(ii) Let G be a group and $Z(G)$ be the center of G. If $\frac{G}{Z(G)}$ is cyclic then show that G is Abelian. 5

7\. *(a)* Let G be a group and H be any subgroup of G. If N is any normal subgroup of G, then show that :

(i) $H \cap N$ is a normal subgroup of H.

(ii) N is a normal subgroup of HN.

(iii) $\frac{HN}{N} \cong \frac{H}{H \cap N}$.

2+2+6=10

(b) Let $f : G \to G'$ be an onto group homomorphism and H be a subgroup of G, H' a subgroup of G'. Prove that :

(i) $f[H]$ is a subgroup of G'.

(ii) $f^{-1}[H']$ is a subgroup of G containing $K = \ker f$, where $f^{-1}[H'] = \{x \in G : f(x) \in H'\}$.

(iii) There exists a one-to-one correspondence between the set of subgroups of G containing K and set of subgroups of G'.

2+3+5=10

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3 (Sem–3/CBCS) MAT HC 3

2023

MATHEMATICS

(Honours Core)

Paper : MAT–HC–3036

(Analytical Geometry)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***all*** the questions : $1 \times 10 = 10$

(a) When the origin is shifted to a point on the x-axis without changing the direction of the axes, the equation of the line $2x + 3y - 6 = 0$ takes the form $lx + my = 0$. What is the new origin ?

(b) Find the centre of the ellipse

$2x^2 + 3y^2 - 4x + 5y + 4 = 0$.

Contd.

(c) Find the angle between the lines represented by the equation

$$x^2 + xy - 6y^2 = 0.$$

(d) Transform the equation $\frac{1}{r} = 1 + \cos\theta$ into cartesian form.

(e) Find the equation of the tangent to the conic $y^2 - xy - 2x^2 - 5y + x - 6 = 0$ at the point $(1, -1)$.

(f) Express the non-symmetric form of equation of a line $\frac{y}{p} + \frac{z}{c} = 1$, $x = 0$ in symmetric form.

(g) Write down the standard form of equation of a system of coaxial spheres.

(h) Write down the equation of a cone whose vertex is origin and the guiding curve is $ax^2 + by^2 + cz^2 = 1$,

$lx + my + nz = p$.

(i) Define a right circular cylinder.

(j) Find the equation of the tangent plane to the conicoid

$$ax^2 + by^2 + cz^2 = 1$$

at the point (α, β, γ) on it.

2. Answer **all** the questions : 2×5=10

(a) If $(at^2, 2at)$ is the one end of a focal chord of the parabola $y^2 = 4ax$, find the other end.

(b) Show that the equation of the lines through the origin, each of which makes an angle α to the line $y = x$ is

$$x^2 - 2xy \sec 2\alpha + y^2 = 0.$$

(c) Find the point where the line

$$\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4}$$

meets the plane $x + y + z = 3$.

(d) Find the equation of the sphere passing the points $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$

(e) Find the equation of the plane which cuts the surface

$2x^2 - 3y^2 + 5z^2 = 1$ in a conic whose centre is $(1, 2, 3)$.

3. Answer ***any four*** questions : 5×4=20

(a) Show that the equation of the tangent to the conic $\frac{l}{r} = 1 + e\cos\theta$ at the point whose vertical angle is α is given by

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha).$$

(b) Prove that the line $lx + my = n$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if

$$\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)}{n^2}.$$

(c) Find the asymptotes of the hyperbola $2x^2 - 3xy - 2y^2 + 3x + y + 8 = 0$ and derive the equations of the principal axes.

(d) Prove that the lines

$\frac{x+5}{3} = \frac{y+4}{1} = \frac{z-7}{-2}$ and

$3x + 2y + z - 2 = 0 = x - 3y + 2z - 13$ are coplanar. Find the equation of the plane in which they lie.

(e) The section of a cone whose guiding curve is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$ by the plane $x = 0$, is a rectangular hyperbola. Prove that the locus of the vertex is $\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1$.

(f) Find the centre and the radius of the circle

$x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0$,

$x - 2y + 2z = 3$.

Answer ***either*** (a) ***or*** (b) from the following questions : 10×4=40

4. (a) (i) Find the point of intersection of the lines represented by the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

(ii) Find the equation of the polar of the point (2, 3) with respect to the conic $x^2 + 3xy + 4y^2 - 5x + 3 = 0$.

5+5=10

(b) (i) Prove that the straight line $y = mx + c$ touches the parabola $y^2 = 4a(x + a)$ if $c = ma + \frac{a}{m}$.

(ii) Find the asymptotes of the hyperbola $xy + ax + by = 0$.

5+5=10

5. (a) Discuss the nature of the conic represented by

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$$

and reduce it to canonical form.

(b) (i) Prove that the sum of the reciprocals of two perpendicular focal chords of a conic is constant.

(ii) Show that the semi-latus rectum of a conic is the harmonic mean between the segments of a focal chord.

5+5=10

6. (a) (i) A variable plane makes intercepts on the co-ordinate axes, the sum of whose squares is a constant and is equal to k^2. Prove that the locus of the foot of the perpendicular from the origin to the plane is

$$\left(x^2 + y^2 + z^2\right)^2\left(x^{-2} + y^{-2} + z^{-2}\right) = k^2$$

(ii) Two spheres of radii r_1 and r_2 intersect orthogonally. Prove that the radius of the common circle is

$$\frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}.$$

5+5=10

(b) Show that the shortest distance between any two opposite edges of the tetrahedron formed by the planes $y + z = 0$, $z + x = 0$, $x + y = 0$, $x + y + z = a$ is $\frac{2a}{\sqrt{6}}$ and that the three lines of shortest distance intersect at the point $x = y = z = -a$.

7. (a) (i) Define reciprocal cone. Show that the cones $ax^2 + by^2 + cz^2 = 0$ and $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$ are reciprocal.

(ii) Find the equation of the right circular cylinder whose guiding curve is $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$.

5+5=10

(b) (i) Find the equation of the director sphere to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(ii) Show that from any point six normals can be drawn to a conicoid $ax^2 + by^2 + cz^2 = 1$.

5+5=10
