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3 (Sem–4/CBCS) MAT HC 1

2023

MATHEMATICS

(Honours Core)

Paper : MAT–HC–4016

(Multivariate Calculus)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : $1\times10=10$

(a) If $f(x, y, z) = x^2ye^{2x} + (x+y-z)^2$, then find $f(-1, 1, -1)$.

(b) If $f(x, y) = sin^{-1}(xy)$, then find f_y at $\left(\frac{1}{2}, \frac{1}{2}\right)$.

(c) Define open disk in R^2.

Contd.

(d) Define critical point of a function f of two variables x and y on an open set.

(e) Let f be a function of two variables x and y defined on the open disk D and $(x_0, y_0) \in D$, then state which one of the following statements is not true :

(i) f is said to have absolute extrema at (x_0, y_0) if $f(x_0, y_0) \geq f(x,y) \; \forall \; (x,y) \in D$.

(ii) f is said to have absolute extrema at (x_0, y_0) if $f(x_0, y_0) \leq f(x,y) \; \forall \; (x,y) \in D$.

(iii) $f(x_0, y_0)$ is a relative maximum if $f(x,y) > f(x_0, y_0) \; \forall \; (x,y) \in D$.

(iv) $f(x_0, y_0)$ is a relative maximum if $f(x,y) \geq f(x_0, y_0) \; \forall \; (x,y) \in D$.

(f) Find the Jacobian of the transformation from Cartesian coordinates to polar coordinates system.

(g) If $f(x,y) = kg(x,y)$ throughout a rectangular region R, then $\iint_R f(x,y)\,dA = k\iint_R g(x,y)\,dA$. State whether this statement is true or false.

(h) Define a vector field.

(i) Find $curl\,\vec{F}$ where $\vec{F}(x, y, z) = xy\hat{i} + yz\hat{j} + zx\hat{k}$.

(j) State when a vector field C is said to be conservative.

2. Answer the following questions : 2×5=10

(a) Find the domain and range of
$$f(x, y) = \frac{1}{\sqrt{9 - x^2 - y^2}}$$

(b) Find the critical point of $f(x, y) = (x-2)^2 + (y-3)^4$ and classify them.

(c) Find $\iint_R (4 - y)\,dA$ where $R = \{(x, y) : 0 \leq x \leq 3,\ 0 \leq y \leq 4\}$.

(d) Find $div\vec{F}$, given that $\vec{F} = \vec{\nabla} f$ where $f(x, y, z) = x^2 y z^3$.

(e) Show that the function $f(x, y) = e^{-x}(\cos y - \sin y)$ is harmonic.

3. Answer **any four** questions : 5×4=20

(a) Find the equation of the tangent plane and the normal line at $P_0(1,-1,2)$ on the surface S given by
$$x^2y + y^2z + z^2x = 5.$$

(b) Use the method of Lagrange's multiplier to maximize $f(x, y) = 16 - x^2 - y^2$ subject to $x + 2y = 6$.

(c) Use polar coordinates to compute the area of the region D bounded above by the line $y = x$ and below by the circle $x^2 + y^2 - 2y = 0$.

(d) Evaluate $\int_C \vec{F}.\,d\vec{R}$ where

$\vec{F} = (y^2 - z^2)\hat{i} + 2yz\,\hat{j} - x^2\hat{k}$ and C is the curve defined parametrically by $x = t^2$; $y = 2t$; $z = t$ for $0 \le t \le 1$.

(e) Verify Stokes' theorem in computing the line integral
$$\int_\Gamma x^2y^3\,dx + dy + z\,dz$$
where Γ is the circle $x^2 + y^2 = a^2$, $z = 0$.

(f) Evaluate $\iiint_B z^2 y e^x \,dv$ where B is the box given by
$$0 \le x \le 1;\ 1 \le y \le 2\ ;\ -1 \le z \le 1$$

4. Answer the following questions : 10×4=40

(a) (i) If $f(x, y, z) = xyz + x^2y^3z^4$, then show that $f_{xyz} = f_{yzx} = f_{zxy}$. 5

(ii) At a certain factory, the daily output is $Q = 60K^{\frac{1}{2}}L^{\frac{1}{3}}$ units, where K denotes the capital investment (in units of Rs. 1,000) and L the size of the labour force (in worker-hours). The current capital investment is Rs. 9,00,000, and 1,000 worker-hours of labour are used each day. Use total differential to estimate the change in output that will result if capital investment is increased by Rs. 1,000 and labour is decreased by 2 worker-hours. 5

OR

(i) Find $\frac{\partial w}{\partial s}$ if $w = 4x + y^2 + z^3$,

where $x = e^{rs^2}$, $y = \log\frac{r+s}{t}$,

$z = rst^2$. 3

(ii) The square $S = \{(x, y) : 0 \le x \le 5, 0 \le y \le 5\}$ is heated in such a way that $T(x, y) = x^2 + y^2$ is the temperature at the point $P(x, y)$. In what direction will heat flow from the point $P_0(3, 4)$? 7

(b) Find all critical points of $f(x, y) = 8x^3 - 24xy + y^3$ and use the 2^{nd} partial derivative test to classify each point as relative extremum or a saddle point.

OR

The function $f(x, y)$ and $g(x, y)$ have continuous first order partial derivatives and f has an extremum at $P_0(x_0, y_0)$ on the smooth constraint curve $g(x, y) = c$. If $\vec{\nabla} g(x_0, y_0) \neq 0$, then show that there is a number λ such that

$\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0)$.

(c) Evaluate $\iiint_D (x^2 + y^2 + z^2)\, dx\, dy\, dz$

where D denotes the region bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = a$; $a > 0$.

OR

(i) Use double integration to find the area bounded by $y = 2 - x$ and $y^2 = 4 - 2x$. 5

(ii) Find the volume of the tetrahedron bounded by the plane $2x + y + 3z = 6$ and the coordinate planes $x = 0$, $y = 0$, $z = 0$. 5

(*d*) State and prove Green's theorem. 2+8=10

OR

Show that the vector field

$$\vec{F} = 2x\left(y^2 + z^2\right)\hat{i} + 2x^2y\,\hat{j} + 3x^2z^2\,\hat{k}$$

is conservative. Also find its scalar potentials. 4+6=10

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3 (Sem–4/CBCS) MAT HC 2

2023

MATHEMATICS

(Honours Core)

Paper : MAT–HC–4026

(Numerical Methods)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : $1\times7=7$

   (a) What is the order of convergence of Regula-Falsi method ?

      (i) 2·312

      (ii) 2·231

      (iii) 1·618

      (iv) 1·321

   (Choose the correct option)

Contd.

(b) Find $\Delta^{n+1} x^n = ?$

(c) Write down Newton's forward interpolation formula.

(d) The Newton-Raphson method is also called as

(i) tangent method

(ii) secant method

(iii) chord method

(iv) diameter method

(Choose the correct option)

(e) In the general Quadrature formula Simpson's one third rule is obtained by putting

(i) $n = 1$

(ii) $n = 2$

(iii) $n = 3$

(iv) $n = 4$

(Choose the correct option)

(f) The value of $\int_0^{\pi/4} \frac{dx}{1+x^2}$ is

(i) 0

(ii) 1

(iii) 2

(iv) None of the above

(Choose the correct option)

(g) Where is Euler's method used ?

2. Answer the following questions : 2×4=8

(a) Define rate of convergence and order of convergence of a sequence.

(b) Evaluate : $\frac{\Delta^2}{E} x^3$

(c) Construct a divided difference table from the following data :

x	-1	1	2	3
y	-21	15	12	3

(d) Why is Lagrange's formula considered to be of more general nature than Newton's formula?

3. Answer ***any three*** questions : $5\times3=15$

(a) What do you mean by algorithm? Use the statistics algorithm to compute the mean and standard deviation of the following data : $1+4=5$

$$1, 3, 5, 7, 9$$

(b) Find a root of the equation

$$x^3-4x-9=0$$

using the bisection method correct up to 3 decimal places.

(c) Show that

(i) $\delta \equiv \nabla(I-\nabla)^{-1/2}$

(ii) $E\Delta \equiv \Delta E$ $3+2=5$

(d) Find the rate of convergence of Newton-Raphson method.

(e) Using Lagrange's interpolation formula for unequal interval, find the values of $f(2)$ and $f(15)$ from the following data :

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

4. Answer the following questions : $10\times3=30$

(a) Determine the root of $xe^x-2=0$ by the method of false position. Perform *five* iterations.

OR

Form an LU decomposition of the following matrix :

$$A=\begin{pmatrix}1 & 4 & 3\\ 2 & 7 & 9\\ 5 & 8 & -2\end{pmatrix}$$

(b) Let $x_0, x_1, \dots x_n$ be $(n+1)$ distinct points on $[a, b]$. If f is continuous on $[a, b]$ and has n continuous derivatives on (a, b), then prove that there exist some $\xi \in (a, b)$ such that

$$f[x_0, x_1, \dots, x_n] = \frac{f^n(\xi)}{\xi\,!}$$

where $f^n(x) = \frac{d^n f(x)}{dx^n}$.

Find the interpolating polynomial from the data given below using divided differences :

x :	-2	0	2
$f(x)$:	4	2	8

5+5=10

OR

Derive the formula for finding first and second order derivatives using Newton's forward difference formula.

Given that

X :	1·0	1·1	1·2	1·3	1·4	1·5	1·6
Y :	7·989	8·403	8·781	9·129	9·451	9·750	10·031

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1 \cdot 1$ 5+5=10

(c) Define numerical integration.

Obtain a general quadrature formula for $\int_a^b f(x)\,dx$.

Hence deduce Simpson's $\frac{1}{3}$rd rule.

1+5+4=10

OR

Write a short note on Euler's method. Give the geometric interpretation of Euler's method.

Give an algebraic interpretation of Euler's method.

Solve by using Euler's method :

$y' = x + y$; $y(0) = 2$ for $0 \le x \le 1$

$h = 0 \cdot 5$ 2+2+2+4=10

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3 (Sem–4/CBCS) MAT HC 3

2023

MATHEMATICS

(Honours Core)

Paper : MAT–HC–4036

(Ring Theory)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : $1 \times 10 = 10$

   (a) Give an example of an infinite noncommutative ring that does not have a unity.

   (b) Define an integral domain.

   (c) What is the characteristic of the ring of 2×2 matrices over integers?

   (d) In an integral domain, if $a \neq 0$ and $ab = ac$, then prove that $b = c$.

Contd.

(e) Show that $2Z \cup 3Z$ is not a subring of Z.

(f) Prove that the correspondence $x \to 5x$ from Z_5 to Z_{10} does not preserve addition.

(g) Characteristic of every field is

(i) 0

(ii) an integer

(iii) either 0 or prime

(iv) either 0 or not prime

(Choose the correct option)

(h) Which of the following is not an integral domain ?

(i) $Z[x]$

(ii) $\{a + b\sqrt{2} : a, b \in Z\}$

(iii) Z_3

(iv) Z_6

(Choose the correct option)

(i) Consider $f(x) = 2x^3 + x^2 + 2x + 2$ and $g(x) = 2x^2 + 2x + 1$ in $Z_3[x]$. Then $f(x) + g(x)$ is

(i) $2x^3 + x$

(ii) $2x^2 + 3x + 3$

(iii) $x^5 + 2$

(iv) $x^5 + 2x^3 + 2$

(Choose the correct option)

(j) The polynomial $f(x) = 2x^2 + 4$ is irreducible over

(i) Q

(ii) $\mathbb{C}$

(iii) $\mathbb{Z}$

(iv) None of the above

(Choose the correct option)

2. Answer the following questions : 2×5=10

(a) Let R be a ring. Prove that $a(-b) = (-a)b = -(ab)$, for all $a, b \in R$.

(b) Prove that the only ideals of a field are $\{0\}$ and F itself.

(c) Show that the ring of integers is an Euclidean domain.

(d) If R is a commutative ring with unity and A is an ideal of R, show that R/A is a commutative ring with unity.

(e) Let $f(x) = x^3 + 2x + 4$ and $g(x) = 3x + 2$ in $Z_5[x]$. Determine the quotient and remainder upon dividing $f(x)$ by $g(x)$.

3. Answer ***any four*** questions of the following : 5×4=20

(a) Prove that

$$Z[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in Z\}$$

is a ring under the ordinary addition and multiplication of real numbers.

(b) (i) If I is an ideal of a ring R such that 1 belongs to I, then show that $I = R$.

(ii) Let R be a ring and $a \in R$. Show that $S = \{r \in R \mid ra = 0\}$ is an ideal of R. 2+3=5

(c) Prove that the ring of integers Z is a principal ideal domain.

(d) Let ϕ be a homomorphism from a ring R to a ring S. If A is a subring of R and B is an ideal of S, prove that

(i) $\phi(A) = \{\phi(a) \mid a \in A\}$ is a subring of S.

(ii) $\phi^{-1}(B) = \{x \in R \mid \phi(x) \in B\}$ is an ideal of R. 2½+2½=5

(e) Let F be a field, $a \in F$ and $f(x) \in F[x]$. Prove that a is a zero of $f(x)$ if and only if $x - a$ is a factor of $f(x)$.

(f) Let F be a field, I a nonzero ideal in $F[x]$, and $g(x)$ an element of $F(x)$. Show that $I = \langle g(x) \rangle$ if and only if $g(x)$ is a nonzero polynomial of minimum degree in I.

*Answer **either** (a) and (b) **or** (c) and (d) of the following questions :* 10×4=40

4. (a) Prove that a finite integral domain is a field. Hence show that for every prime p, Z_p, the ring of integers modulo p, is a field. 4+2=6

Contd.

(b) Show that $\frac{R[x]}{\langle x^2+1 \rangle}$ is a field. 4

OR

(c) Prove that every field is an integral domain. Is the converse true? Justify with an example. 2+1=3

(d) Define prime ideal and maximal ideal of a ring. Show that $\langle x \rangle$ is a prime ideal of $Z[x]$ but not a maximal ideal of it. 2+5=7

5. (a) Let ϕ be a homomorphism from a ring R to a ring S. Prove that ϕ is an isomorphism if and only if ϕ is onto and $ker\phi = \{r \in R \mid \phi(r) = 0\} = \{0\}$. 5

(b) If ϕ is an isomorphism from a ring R to a ring S, then show that ϕ^{-1} is an isomorphism from S to R. 5

OR

(c) Let R be a ring with unity e. Show that the mapping $\phi : \mathbb{Z} \to R$ given by $n \to ne$ is a ring homomorphism. 5

(d) Define kernel of a ring homomorphism. Let ϕ be a homomorphism from a ring R to a ring S. Prove that $ker\phi$ is an ideal at R. 1+4=5

6. (a) State and prove the second isomorphism theorem for rings. 1+7=8

(b) Let R be a commutative ring of characteristic 2. Show that the mapping $a \to a^2$ is a ring homomorphism from R to R. 2

OR

(c) State and prove the third isomorphism theorem for rings. 1+6=7

(d) Prove that every ideal of a ring R is the kernel of a ring homomorphism of R. 3

7. (a) Let F be a field. If $f(x) \in F[x]$ and $deg\, f(x) = 2$ or 3, then prove that $f(x)$ is reducible over F if and only if $f(x)$ has a zero in F. 4

(b) In a principal ideal domain prove that an element is an irreducible if and only if it is a prime. 6

OR

(c) Let p be a prime and suppose that $f(x) \in Z[x]$ with $deg\, f(x) \geq 1$. Let $\overline{f(x)}$ be the polynomial in $Z_p[x]$ obtained from $f(x)$ by reducing all the coefficients of $f(x)$ modulo p. If $f(x)$ is irreducible over Z_p and $deg\, \overline{f(x)} = deg\, f(x)$, then prove that $f(x)$ is irreducible over Q. 5

(d) Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0 \in Z[x].$$

If there is a prime p such that

$p \nmid a_n$, $p \mid a_{n-1}, ..., p \mid a_0$ and $p^2 \nmid a_0$,

then prove that $f(x)$ is irreducible over Q. 5
