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3 (Sem–2/CBCS) MAT HC 1

2023

MATHEMATICS

(Honours Core)

Paper : MAT–HC–2016

(Real Analysis)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : $1\times10=10$

(a) Give an example of a set which is not bounded below.

(b) Write the completeness property of $\mathbb{R}$.

(c) If $S=\left\{\frac{1}{n} : n\in\mathbb{N}\right\}$, then what will be $\inf S$?

Contd.

(d) The unit interval $[0,1]$ in $\mathbb{R}$ is not countable.

(State whether True or False)

(e) Define a convergent sequence of real numbers.

(f) What is the limit of the sequence. $\{x_n\}$,

where $x_n = \dfrac{5n+2}{n+1}$, $n \in \mathbb{N}$?

(g) A bounded monotone sequence of real numbers is convergent.

(State whether True or False)

(h) What is the value of r if the geometric series $\sum_{n=0}^{\infty} r^n$ is convergent ?

(i) The series $1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\ldots$ is not convergent.

(State whether True or False)

(j) If $\sum_{n=1}^{\infty} u_n$ is a positive term series such that $\lim_{n\to\infty} (u_n)^{1/n} = l$, then the series converges, if

(i) $l<1$

(ii) $0<l<2$

(iii) $l>1$

(iv) $1 \le l < 2$

(Choose the correct option)

2. Answer the following questions : 2×5=10

(a) Find the supremum of the set
$S = \{x \in \mathbb{R} : x^2 - 3x + 2 < 0\}$.

(b) If (x_n) and (y_n) are convergent sequences of real numbers and $x_n \le y_n \ \forall\, n \in \mathbb{N}$, then show that
$\lim_{n\to\infty} x_n \le \lim_{n\to\infty} y_n$.

(c) Show that the sequence $\left((-1)^n\right)$ is divergent.

(d) Define absolutely convergent series and give an example.

(e) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ is convergent.

3. Answer ***any four*** questions : 5×4=20

(a) Prove that if $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ such that $x \leq n_x$.

(b) If x and y are real numbers with $x < y$, then show that there exists an irrational number z such that $x < z < y$.

(c) Show that if a sequence (x_n) of real numbers converges to a real number x, then any subsequence of (x_n) also converges to x.

(d) Show that the sequence

$$\left((-1)^n + \frac{1}{n}\right), \quad n \in \mathbb{N}$$ is not a Cauchy sequence.

(e) Using ratio test establish the convergence or divergence of the series whose nth term is $\frac{n!}{n^n}$.

(f) Let $z = (z_n)$ be a decreasing sequence of strictly positive numbers with $lim(z_n) = 0$. Prove that the alternating series $\Sigma(-1)^{n+1} z_n$ is convergent.

4. Answer the following questions : 10×4=40

(a) Prove that the set $\mathbb{R}$ of real numbers is not countable.

Or

If S is a subset of $\mathbb{R}$ that contains at least two points and has the property : if $x, y \in S$ and $x < y$, then $[x, y] \subseteq S$, then show that S is an interval.

(b) Prove that a sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Or

Let (x_n) be a sequence of positive real numbers such that $L = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$ exists. If $L < 1$, then show that (x_n) converges and $\lim_{n\to\infty} x_n = 0$.

(c) (i) Show that $\lim \left(\frac{1}{n^2+1}\right) = 0$ 2½

(ii) Show that the sequence $\left(\frac{1}{n}\right)$ is a Cauchy sequence. 2½

(iii) Prove that every contractive sequence is a Cauchy sequence. 5

Or

State and prove the monotone subsequence theorem. 10

(d) Prove that a positive term series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $0 < p \le 1$.

Or

Show that a necessary condition for convergence of an infinite series $\sum_{n=1}^{\infty} u_n$ is that $\lim_{n\to\infty} u_n = 0$. Demonstrate by an example that this is not a sufficient condition for the convergence.

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3 (Sem–2/CBCS) MAT HC 2

2023

MATHEMATICS

(Honours Core)

Paper : MAT–HC–2026

(Differential Equation)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×7=7

   (a) What is meant by implicit solution of a differential equation ?

   (b) Find the Wronskian of the set $\{e^x, e^{-x}\}$.

   (c) Determine whether the differential equation $2xy\,dx + (1+x^2)\,dy = 0$ is exact.

Contd.

(d) Determine the integrating factor of the following differential equation :

$$x^4 \frac{dy}{dx} + 2x^3 y = 1$$

(e) State the process in which compartmental model technique is used to formulate the mathematical model.

(f) What do you mean by a singular solution of a differential equation?

(g) Write down the condition under which the n solutions $f_1, f_2, \ldots, f_n$ of an n^{th} order homogenous linear differential equation are linearly independent on $a \le x \le b$.

2. Answer the following questions : 2×4=8

(a) Determine whether the pair of function $f(x) = e^x \sin x$ and $g(x) = e^x \cos x$ are linearly independent or linearly dependent on the real line.

(b) State the assumption made in developing a model of radioactivity. Draw the input-output compartmental diagram for radioactive nuclei.

(c) Find the general solution of

$$2y'' - 7y' + 3y = 0$$

(d) Reduce the Bernoulli equation

$$x \, dy/dx + 6y = 3x^{4/3}$$

to linear equation by appropriate transformation.

3. Answer the following questions : ***(any three)*** 5×3=15

(a) Solve by the method of variation of parameter

$$y'' + y = \tan x$$

(b) Solve the initial value problem

$$(x+2)\frac{dy}{dx} + y = f(x)$$

where

$$f(x) = \begin{cases} 2x & 0 \le x \le 2 \\ 4 & x > 2 \end{cases} \qquad y(0) = 4$$

(c) It has been observed that in a population following the limited growth with harvesting model, the growth rate is 1, carrying capacity is 10, and the constant rate of harvesting is 0·9. If the initial population is x_0, find the population after time t.

(d) Find the general solution of

$$y^{(3)} - 6y'' + 11y' - 6y = 2xe^x$$

(e) Consider the differential equation

$$(3y + 4xy^2)\,dx + (2x + 3x^2y)\,dy = 0$$

(i) Show that this equation is not exact.

(ii) Find the integrating factor of the differential equation and hence solve it.

1+4=5

4. Answer the following questions : ***(any three)***

10×3=30

(a) Suppose that the functions $M(x, y)$ and $N(x, y)$ are continuous and have continuous first-order partial derivatives in the region R in xy-plane. Prove that the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

at each point of R.

Or

Solve the differential equation by making suitable transformation

$$(x - 2y + 1)\,dx + (4x - 3y - 6)\,dy = 0$$

(b) In a fish farm, fish are harvested at a constant rate of 2100 fish per week. The per capita death rate for the fish is 0·2 fish per day per fish and the per capita birth rate is 0·7 fish per day per fish.

(i) Write a word equation describing the rate of change of the fish population. Hence obtain a differential equation for the population of fish.

(ii) If the fish population at a given term is 2,40,000; give an estimate of the number of fish born in one week.

(iii) Determine if there are any value for which the fish population is in equilibrium.

2+3+5=10

Or

(i) A culture initially has N_0 number of bacteria. At $t = 1$ hour the number of bacteria is measured to be $\frac{3}{2}N_0$. If the rate of growth is proportional to the number of bacteria present, determine the time necessary for the number of bacteria to triple. 5

(ii) The differential equation $\frac{dC}{dt} = \frac{F}{V}(C_{in} - C)$ describes the level of pollution in a lake, where V is the volume of the lake, F is the flow, C is the concentration of pollution at time t and C_{in} is the concentration of pollution entering the lake.

(i) Solve the differential equation with the initial condition $C(0) = C_0$.

(ii) How long will it take for the lake's pollution level to reach 5% of its initial level if only fresh water flows into the lake?

3+2=5

(c) (i) Find the general solution of

$$y^{(3)} + y'' = 3e^{x} + 4x^2$$

(ii) Solve the Euler equation

$$x^2y'' + xy' + 9y = 0$$

5+5=10

Or

Solve the initial value problem

$y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x$

given $y(0) = 1,\ y'(0) = 2$. 10