Total number of printed pages–28

3 (Sem–6/CBCS) MAT HE 1/2/3/4

2023

MATHEMATICS

(Honours Elective)

Answer the Questions from any one Option.

OPTION – A

(Boolean Algebra and Automata Theory)

Paper : MAT–HE–6016

Full Marks : 80

Time : Three hours

OPTION – B

(Biomathematics)

Paper : MAT–HE–6026

Full Marks : 80

Time : Three hours

OPTION – C

(Mathematical Modeling)

Paper : MAT–HE–6036

Full Marks : 60

Time : Three hours

OPTION – D

(Hydromechanics)

Paper : MAT–HE–6046

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

Contd.

OPTION–A

(*Boolean Algebra and Automata Theory*)

Paper : MAT–HE–6016

1. Answer the following questions : 1×10=10

(a) A relation $\leq$ on a set P is called quasi-order, if

(i) reflexive, transitive and antisymmetric

(ii) reflexive and antisymmetric

(iii) transitive and antisymmetric

(iv) None of the above

(Choose the correct answer)

(b) An ordered set P is an antichain if _______ in P only if _______.

(Fill in the blanks)

(c) Let P^D be the dual of any ordered set P. Then

(i) $x \leq y$ holds in P^D if $x \leq y$ holds in P

(ii) $x \leq y$ holds in P^D if $y \leq x$ holds in P

(iii) $x \leq y$ holds in P^D if $x = y$ holds in P

(iv) None of the above

(Choose the correct answer)

(d) Define lattice homomorphism.

(e) Let L be a lattice and $a, b \in L$. If $a \leq b$, then

(i) $a \vee b = b, a \wedge b = a$

(ii) $a \vee b = b$ but not $a \wedge b = a$

(iii) $a \wedge b = a$ but not $a \vee b = b$

(iv) None of the above

(Choose the correct answer)

(f) Define conjunctive normal form.

(g) For all x, y in a Boolean algebra,

(i) $(x \wedge y)' = x' \vee y'$ and $(x \vee y)' = x' \wedge y'$

(ii) $(x \wedge y)' = x' \wedge y'$ and $(x \vee y)' = x' \vee y'$

(iii) $(x \wedge y)' = y'$ and $(x \vee y)' = x'$

(iv) None of the above

(Choose the correct answer)

(h) Define Boolean polynomial function.

(i) What is the empty string?

(j) Define closure properties of regular languages.

2. Answer the following questions : 2×5=10

(a) Prove that the elements of any arbitrary lattice satisfy the following inequalities :

(i) $x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z)$

(ii) $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$

(b) Prove that every chain is a distributive lattice.

(c) Define NFA.

(d) Define atom. Prove that every atom of a lattice with zero is join-irreducible.

(e) Prove that if L and M are regular languages, then $L \cup M$ is also a regular language.

3. Answer ***any four*** questions from the following : 5×4=20

(a) (i) Prove that two finite ordered set P and Q are order-isomorphic if and only if they can be drawn with identical diagrams.

(ii) Define monomorphism. Let f be a monomorphism from the lattice L into the lattice M. Show that L is isomorphic to a sublattice M.

(b) (i) Let C_1 and C_2 be the finite chains $\{0, 1, 2\}$ and $\{0, 1\}$ respectively. Draw the Hasse diagram of the product lattice $C_1 \times C_2 \times C_3$.

(ii) Let L be a distributive lattice with 0 and 1. Prove that if a has a complement a', then $a \vee (a' \wedge b) = a \vee b$.

(c) (i) State and prove De Morgan's laws of a Boolean algebra.

(ii) Let $f : B_1 \to B_2$ be a Boolean homomorphism. Then prove the following :

(1) $f(0) = 0,\ f(1) = 1$

(2) For all $x, y \in B_1$

$$x \leq y \Rightarrow f(x) \leq f(y).$$

(d) Let $p, q \in P_n$; $p \sim q$ and let B be an arbitrary Boolean algebra. Then, prove that $\overline{p}_B = \overline{q}_B$.

(e) Prove that a language L is accepted by some DFA if and only if L is accepted by some NFA.

(f) Prove that every regular language is a context-free language.

4. Answer the following questions : 10×4=40

(a) (i) Let P and Q be finite ordered sets and let $f : P \to Q$ be a bijective map. Then, prove that the following are equivalent :

(1) f is an order-isomorphism;

(2) $x < y$ in P if and only if $f(x) < f(y)$ in Q;

(3) $x \prec y$ in P if and only if $f(x) \prec f(y)$ in Q. 5

(ii) Let P be an ordered set. Then, prove that

$\mathcal{O}(P \oplus 1) \cong \mathcal{O}(P) \oplus 1$ and

$\mathcal{O}(1 \oplus P) \cong \oplus 1\, \mathcal{O}(P)$ 5

OR

Let P be a finite ordered set.

(i) Show that $Q = \downarrow Max\, Q$, for all $Q \in \mathcal{O}(P)$

(ii) Establish a one-to-one correspondence between the elements of $\mathcal{O}(P)$ and antichains in P

(iii) Hence show that for all $x \in P$,

$$|\mathcal{O}(P)| = |\mathcal{O}(P \setminus \{x\})| + |\mathcal{O}(P \setminus (\downarrow x U \uparrow x))|$$

10

(b) (i) Let L be a distributive lattice and let $p \in L$ be join-irreducible with $p \le a \vee b$. Then, prove that $p \le a$ or $p \le b$. 5

(ii) Prove that generalized distributive inequality for lattices

$$y \wedge \left(\bigvee_{i=1}^{n} x_i \right) \ge \bigvee_{i=1}^{n} (y \wedge x_i).$$ 5

OR

(iii) Let B be a Boolean algebra. Then, prove that the set $P_n(B)$ is a Boolean algebra and subalgebra of the Boolean algebra $F_n(B)$ of all functions from B_n into B. 5

(iv) Find the DNF of

$$x_1(x_2+x_3)' + (x_1x_2+x_3')x_1$$ 5

(c) *(i)* Prove that a polynomial $p \in P_n$ is equivalent to the sum of all prime implications of p. 5

(ii) Find three prime implications of $xy + xy'z + x'y'z$. 5

OR

(iii) Determine the symbolic representation of the circuit given by

$$p = (x_1+x_2+x_3)(x_1'+x_2)(x_1x_3+x_1'x_2)(x_2'+x_3)$$

5

(iv) Design a switching circuit that enables you to operate one lamp in a room from four different switches in that room. 5

(d) *(i)* If L, M and N are any languages, then prove that

$$L(M \cup N) = LM \cup LN.$$ 5

(ii) If L is a regular language over alphabet Σ, then $\overline{L} = \Sigma^* - L$ is also a regular language. 5

OR

(iii) Consider the CFG K defined by productions

$$S \rightarrow aSbS \mid bSaS \mid \varepsilon$$

Prove that $L(K)$ is the set of all strings with an equal number of a's and b's. 5

(iv) Let $G = (V, T, P, S)$ be a CFG, and suppose that there is a derivation

$A \overset{*}{\underset{G}{\Rightarrow}} w$, where w is in T^*. Then, prove that the recursive inference procedure applied to G determines that w is in the language of variable A. 5

OPTION–B

(Biomathematics)

Paper : MAT–HE–6026

1. Answer the following questions : 1×10=10

(a) What is an autonomous system?

(b) The zero equilibrium/positive equilibrium is often not a desired state in biological system.

(Choose the correct answer)

(c) Write a difference between continuous growth and discrete growth.

(d) Give an example of nonlinear, autonomous second order difference equation.

(e) Write *one* use of Routh-Hurwitz criteria.

(f) Equilibria are also known as

(a) steady state

(b) fixed points

(c) critical points

(d) All of the above

(Choose the correct answer)

(g) Write the condition that a first order partial derivative of a system is locally asymptotically stable.

(h) Write the condition that the equilibrium $\bar{x}$ of $\frac{dx}{dt} = f(x)$ is hyperbolic.

(i) Write the three population classes in Kermack-McKendrick model.

(j) Define a characteristic polynomial for second order equation.

2. Answer the following questions : 2×5=10

(a) Define a difference equation of order k.

(b) State Frobenius theorem.

(c) Distinguish between local stability and global stability.

(d) Consider the linear differential equation

$$\frac{d^3x}{dt^3} + 4\frac{d^2x}{dt^2} + \frac{dx}{dt} + ax = 0$$

Show that its solution approaches zero.

(e) For the linear differential equation $\frac{dx}{dt} = AX$, the matrix A is given by

$$A = \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}.$$ Find the eigenvalues.

3. Answer **any four** questions : 5×4=20

(a) The difference equation is given by $x_{t+4} + ax_t = 0$.

Find its characteristic equation and its solutions.

(b) Find the eigenvalues and eigenvectors of matrix A when

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

Then find the general solutions to $x(t+1) = Ax(t)$.

(c) Find all the equilibria for the difference equation $x_{t+1} = ax_t \exp(-rx_t)$, $a, r > 0$.

(d) Consider the differential equation

$$x'''(t) - 4x''(t) = 0$$

where $x'' = \frac{d^2x}{dt^2}$ and so on.

Find its characteristic equation and its roots or eigenvalues and verify that the solutions are linearly independent or not.

(e) A mathematical model for the growth of a population is

$$\frac{dx}{dt} = \frac{2x^2}{1+x^4} - x = f(x),\ x(0) \geq 0$$

where x is the population density. Find the equilibria and determine their stability.

(f) Suppose an SIS epidemic model with disease-related deaths and a growing population satisfies

$$\frac{dN}{dt} = N(b - CN) - \alpha I,\ b, c, \alpha > 0$$

(i) Find the differential equations satisfied by the proportions

$$i(t) = \frac{I(t)}{N(t)} \text{ and } s(t) = \frac{S(t)}{N(t)}$$

Then find the basic reproduction number.

(ii) Do the dynamics of $N(t)$ change with disease? Is it possible for $N(t) \to 0$? Note that $m(N) = CN$

and $\frac{dN}{dt} = N(b - CN - \alpha i)$.

4. Answer the following questions : 10×4=40

(a) Find the general solution to the non-homogeneous linear difference equation

$x_{t+2} + x_{t+1} = 6x_t = 5$

Or

Suppose the Leslie matrix is given by

$$L = \begin{pmatrix} 0 & \frac{3a^2}{2} & \frac{3a^3}{2} \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}, \ a > 0$$

(i) Find the characteristic equation, eigenvalues and inherent net reproduction number R_0 of L.

(ii) Show that L is primitive.

(iii) Find the stable age distribution.

(b) The following epidemic model is referred to as an SIS epidemic model. Infected individuals recover but do not become immune. They become immediately susceptible again.

$$S_{t+1} = S_t - \frac{\beta}{N} I_t S_t + (\gamma + b) I_t$$

$$I_{t+1} = I_t (1 - \gamma - b) + \frac{\beta}{N} I_t S_t$$

Assume that $0 < \beta < 1, 0 < b + \gamma < 1$

$S_0 + I_0 = N$ and $S_0, I_0 > 0$

(i) Show that $S_t + I_t = N$ for $t = 1, 2, \ldots$.

(ii) Show that there exist two equilibria and they are both non-negative if $R_0 = \frac{\beta}{b + \gamma} \geq 1$.

Or

Discuss a predator-prey model with a suitable example by finding its equilibria, local stability and global stability.

(c) State briefly a measles model with vaccination.

Or

Show that the solution to the pharmacokinetics model is

$$x(t) = \frac{1}{a}\left(1 - e^{-at}\right)$$

$$y(t) = \frac{1}{b} + \frac{e^{-at}}{a-b} - \frac{ae^{-bt}}{b(a-b)}$$

(d) For the following differential equation, find the equilibria, then graph the phaseline diagram. Use the phaseline diagram to determine the stability of equilibrium

$$\frac{dx}{dt} = x(a-x)(x-b)^2, \; 0 < a < b.$$

Or

Discuss briefly about simple Kermack-McKendric epidemic model.

OPTION–C

(Mathematical Modeling)

Paper : MAT–HE–6036

1. Answer the following questions : 1×7=7

(a) Write Legendre's equation of order *n*.

(b) When does a power series converge if f be the radius of convergence and $0 < \rho < \infty$?

(c) Write the value of $\Gamma 3$.

(d) Find the Laplace transform of $F(t) = 1$.

(e) Monte Carlo simulation is a probabilistic/logistic model.

(Choose the correct answer)

(f) The linear congruence method was introduced by ________ .

(Fill in the blank)

(g) Which one is not a high level simulation language?

(i) GPSS

(ii) SPSS

(iii) SIMAN

(iv) DYNAMO

(Choose the correct answer)

2. Answer the following questions : 2×4=8

(a) Show that $\Gamma(x+1) = x\Gamma(x)$.

(b) Find the inverse Laplace transform of

$$F(s) = \frac{1}{s(s-3)}.$$

(c) Write *two* advantages of Monte Carlo simulation.

(d) Why is sensitivity analysis important in linear programming?

3. Answer ***any three*** questions of the following : 5×3=15

(a) Solve the equation

$$y' + 2y = 0$$

(b) Find the exponents in the possible Frobenius series solutions of the equation

$$2x^2(1+x)y'' + 3x(1+x)^3 y' - (1-x^2)y = 0$$

(c) Suppose that m is a positive integer. Show that

$$\Gamma\left(m+\tfrac{2}{3}\right) = \frac{2.5.8....(3m-1)}{3^m}\Gamma\left(\frac{2}{3}\right).$$

(d) Solve the equation

$$4x^2y'' + 8xy' + (x^4 - 3)y = 0$$

(e) Write briefly about different steps of the simplex method.

4. Answer the following : 10×3=30

(a) Solve the initial value problem

$$\left(t^2-2t-3\right)\frac{d^2y}{dt^2}+3(t-1)\frac{dy}{dt}+y=0;$$

$y(1)=4,\ \ y'(1)=-1$

Or

Find the Frobenius series solutions of $xy''+2y'+xy=0$.

(b) Using Monte Carlo simulation, write an algorithm to calculate an approximation to π by considering the number of random points selected inside the quarter circle.

$$Q: x^2+y^2=1,\ \ x\geq 0,\ y\geq 0$$

where the quarter circle is taken to be inside the square $S: 0\leq x\leq 1$ and $0\leq y\leq 1$.

Or

Solve the equation $y''+y=0$.

(c) Write briefly about middle square method.

Or

A small harbor has unloading facilities for ships. Only one ship can be unloaded at any time. The unloading time required for a ship depends on the type and amount of cargo and varies from 45 to 90 minutes.

Below is given a situation with 5 ships :

	Ship 1	Ship 2	Ship 3	Ship 4	Ship 5
Time between successive ships : (in minutes)	30	15	20	25	120
Unloading time :	40	35	60	45	75

(i) Draw the time line diagram depicting clearly the situation for each ship, the idle time for the harbor and the waiting time.

(ii) List the waiting time for all the ships and find the average waiting time.

OPTION–D

(Hydromechanics)

Paper : MAT–HE–6046

1. Answer the following questions : 1×10=10

(a) What happens when there is an increase of pressure at any point of a liquid at rest under given external forces?

(b) State Charles' law.

(c) What is internal energy?

(d) Define adiabatic expansion.

(e) Give an example of application of atmospheric pressure in daily life.

(f) Define ideal fluid.

(g) Potential flow is the .... flow of an inviscid or perfect flow.

(Fill in the gap)

(h) Equation of continuity by Euler's method is

(i) $\frac{\partial\rho}{\partial t} + \rho\nabla\cdot\vec{a} = 0$

(ii) $\frac{\partial\rho}{\partial t} - \rho\nabla\cdot\vec{a} = 0$

(iii) $\frac{\partial\rho}{\partial t} - \nabla\cdot(\rho\vec{a}) = 0$

(iv) None of the above

(Choose the correct option)

(i) Streamlines and pathlines become the same when the motion is ....

(Fill in the gap)

(j) Velocity potential φ satisfies which of the following equations?

(i) Bernoulli

(ii) Cauchy

(iii) Laplace

(iv) None of the above

(Choose the correct option)

2. Answer the following questions : 2×5=10

(a) Show that the surfaces of equal pressure are intersected orthogonally by the lines of force.

(b) Define field of force and line of force with examples.

(c) If ρ_0 and ρ be the densities of a gas at 0° and t° Centigrade respectively, then establish the relation $\rho_0 = \rho(1+\alpha t)$

where $\alpha = \frac{1}{273}$.

(d) Distinguish between the streamlines and pathlines.

(e) Give examples of irrotational and rotational flows.

3. Answer the following questions : ***(any four)*** 5×4=20

(a) Determine the necessary condition that must be satisfied by a given distribution of forces X, Y, Z, so that the fluid may maintain equilibrium.

(b) A quadrant of a circle is just immersed vertically with one edge in the surface, in a liquid, the density of which varies as the depth. Determine the centre of pressure (C.P.).

(c) A box is filled with a heavy gas at a uniform temperature. Prove that if a is the altitude of the highest point above the lowest and p and p' are the pressures at these two points, the ratio of the pressure to the density at any point is equal to

$$\frac{ag}{\log p'/p}.$$

(d) If w is the area of cross-section of a stream filament, prove that the equation of continuity is

$$\frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial s}(\rho w q) = 0$$

where δs is an element of arc of the filament in the direction of flow and q is the speed.

(e) Determine the acceleration of a fluid particle when velocity distribution is $\vec{a} = \hat{i}(Ax^2yt) + \hat{j}(By^2zt) + \hat{k}(Czt^2)$ where A, B, C are constants. Also find the velocity components.

(f) The velocity field at a point in fluid is given by $\vec{a} = (x/t, y, 0)$. Obtain the pathlines.

4. Answer the following questions : 10×4=40

(a) A mass of homogeneous liquid contained in a vessel revolves uniformly about a vertical axis. You are required to determine the pressure at any point and the surfaces of equal pressure.

OR

A mass m of elastic fluid is rotating about an axis with uniform angular velocity ω, and is acted on by an attraction towards a point in that axis equal to μ times the distance, μ being greater than ω^2. Prove that the equation of a surface of equal density ρ is

$$\mu(x^2 + y^2 + z^2) - \omega^2(x^2 + y^2) = k \log\left\{\frac{\mu(\mu - \omega^2)^2}{8\pi^3} \cdot \frac{m^2}{\rho^2 k^3}\right\}.$$

(b) A hemispherical bowl is filled with water and two vertical planes are drawn through its central radius, cutting off a semi-lune of the surface. If 2α be the angle between the planes, prove that the angle which the resultant pressure on the surface makes with the vertical

$$= \tan^{-1}\left(\frac{\sin\alpha}{\alpha}\right).$$

OR

A gaseous atmosphere in equilibrium is such that $p = k\rho^\gamma = R\rho T$ where p, ρ, T are the pressure, density and temperature and k, γ, R are constants. Prove that the temperature decreases upwards at a constant rate α, so that $\frac{dT}{dZ} = -\alpha = -\frac{g}{R} \cdot \frac{\gamma - 1}{\gamma}$. In a certain atmosphere of uniform composition $T = T_0 = \beta z$ where T_0 and β are constants and $\beta < \alpha$. Find the pressure and density and show that they both vanish at height $\frac{T_0}{\beta}$.

(c) Derive the equation of continuity in Cartesian coordinates. Also what happen, if the fluid is homogeneous and incompressible.

OR

Derive the equation of continuity by the Lagrangian method.

(d) The velocity components for a two-dimensional fluid system can be given in Eulerian system by

$$U = 2x + 2y + 3t$$

$$V = x + y + \frac{t}{2}$$

Find the displacement of a fluid particle in the Lagrangian system.

OR

Obtain Euler's equation of motion of a non-viscous fluid in the form

$$\frac{D\vec{q}}{Dt} = \vec{F} - \frac{1}{\rho}\nabla P.$$

Total number of printed pages–20

3 (Sem–6/CBCS) MAT HE 5/6/7

2023

MATHEMATICS

(Honours Elective)

Answer the Questions from any one Option.

OPTION–E

(Rigid Dynamics)

Paper : MAT–HE–6056

OPTION–F

(Group Theory–II)

Paper : MAT–HE–6066

OPTION–G

(Mathematical Finance)

Paper : MAT–HE–6076

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

Contd.

OPTION–E

(Rigid Dynamics)

Paper : MAT–HE–6056

1. Answer the following questions : 1×10=10

(a) The distance between *any two* particles of a rigid body always varies. *(State True **or** False)*

(b) Define product of inertia of a body.

(c) When are *two* systems said to be equimomental ?

(d) Define the momental ellipsoid.

(e) Define centroid of a system of particles.

(f) Write the principle of conservation of linear momentum under finite forces.

(g) State the parallel axes theorem.

(h) What are generalized co-ordinates ?

(i) Define angular momentum of a system of particles.

(j) Define centre of oscillation.

2. Answer the following questions : 2×5=10

(a) Write a very short note on simple equivalent pendulum.

(b) Write the necessary and sufficient conditions for two systems to be equimomental.

(c) Obtain the scaler equations of motion of a particle of mass M, placed at the centre of inertia of the body and acted on by the forces $\Sigma X, \Sigma Y, \Sigma Z$ parallel and equal to the external forces acting on different points of the body.

(d) A rigid body consists of three particles of masses 2 units, 3 units and 5 units located at the points (1, 1, 0), (–1, 0, 1) and (0, 1, 1) respectively. Find the products of inertia about

(i) x axis and y axis;

(ii) y axis and z axis.

(e) A rigid body of mass 5 units rotates with angular velocity $\vec{\omega} = (1, 1, -1)$ and has the angular momentum $\vec{\Omega} = (4, 2, 1)$. Find the kinetic energy of the body.

Contd.

3. Answer ***any four*** questions : 5×4=20

(a) Show that the momental ellipsoid at the centre of an ellipsoid is $\left(b^2+c^2\right)x^2+\left(a^2+c^2\right)y^2+\left(a^2+b^2\right)z^2=$ constant.

(b) If α, β, γ and h be the distances of the vertices and the centre of inertia of a uniform triangular lamina of mass m from any line, prove that the moment of inertia about that line is $\frac{1}{12}m\left[\left(\alpha^2+\beta^2+\gamma^2\right)+9h^2\right]$

(c) Derive the general vector equations of motion of a rigid body mentioning the name of the principle used in the derivation.

(d) Find the time of complete oscillation of a compound pendulum consisting of a rod of mass m and length a carrying at one end a sphere of mass m and diameter $2b$, the other end of the rod being fixed.

(e) Prove that if a rigid body be moving under the action of external forces the sum of whose moments about a given line is zero throughout the motion, the angular momentum of the body about that line remains unchanged throughout the motion.

(f) When a body moves under the action of a system of conservative forces, the sum of its kinetic and potential energies is constant throughout the motion. Prove the statement.

4. Answer the following : 10×4=40

(a) *(i)* Show that the moment of inertia of semi-circular lamina about a tangent parallel to the bounding diameter is $Ma^2\left(\frac{5}{4}-\frac{8}{3\pi}\right)$, where a is the radius and M is the mass of the lamina. 5

(ii) Given moments and products of inertia about axes through the centre of gravity, find the moments and products of inertia about parallel axis. 5

Or

(a) (i) A lamina in the form of an ellipse is rotating in its own plane about one of its foci with angular velocity ω. This focus is set free and the other focus at the same instant is fixed. Show that the ellipse now rotates about it with angular velocity $\left(\frac{2-5e^2}{2+3e^2}\right)\omega$. 5

(ii) The lengths AB and AD of the sides of a rectangle $ABCD$ are $2a$ and $2b$ respectively. Show that the inclination to AB of one of the principal axes at A is

$$\frac{1}{2}\tan^{-1}\frac{3ab}{2\left(a^2-b^2\right)}.$$ 5

(b) (i) State and prove the D'Alembert's principle. 5

(ii) Find the length of the simple equivalent pendulum for an elliptic lamina when the axis is a latus rectum. 5

Or

(b) (i) Use Lagrange's equations to find the equation of motion of the compound pendulum which oscillates in a vertical plane about a fixed horizontal axis. 5

(ii) An elliptic lamina is rotating about its centre on a smooth horizontal table. If ω_1, ω_2, ω_3 be its angular velocities when the extremity of its axis, its focus and the extremity of its minor axis respectively become fixed, prove that

$$\frac{7}{\omega_1}=\frac{6}{\omega_2}+\frac{5}{\omega_3}.$$ 5

(c) (i) If $S \equiv Ax^2+By^2+Cz^2-2Dyz-2Ezx-2Fxy=$ constant be the equation of the momental ellipsoid at the centre of gravity O of a body referred to any rectangular axes through O, then prove that momental ellipsoid at the point (p, q, r) is

$$S+M\left[(qz-ry)^2+(rx-pz)^2+(py-qx)^2\right]=$$

constant, where M is the mass of the body. 5

(ii) A rough uniform board of mass m and length $2a$ rests on a smooth horizontal plane and a man of mass M walks on it from one end to the other. Show that the distance through which the board moves in this time is

$$\frac{2Ma}{M+m}.$$ 5

Or

(c) (i) A uniform rod of mass m and length $2a$ can turn freely about one end which is fixed. It is started with angular velocity ω from the position in which it hangs vertically. Find its angular velocity at any instant. 5

(ii) A uniform elliptic board swings about a horizontal axis at right angle to the board and passing through one focus. If the centre of oscillation be at the other focus, prove that the eccentricity of the ellipse is $\sqrt{2/5}$. 5

(d) A uniform solid sphere rolls down an inclined plane whose inclination to the horizontal is α. Show that the least co-efficient of friction between it and the plane, so that it may roll and not slide is $\frac{2}{7}\tan\alpha$. 10

Or

Derive Lagrange's equation in generalized co-ordinates.

OPTION–F

(Group Theory–II)

Paper : MAT–HE–6066

1. Answer the following as directed : 1×10=10

(a) An isomorphim from a group to itself is called ·

(i) endomorphism

(ii) monomorphism

(iii) automorphism

(iv) None of the above

(Choose the correct option)

(b) A group of order 4 is isomorphic to z_4 or $z_2 \oplus z_2$. Is it true?

(c) Define internal direct product of *two* subgroups of a group.

(d) List the elements of $\cup(8) \oplus \cup(10)$.

(e) Let G be a group. $|Inn(G)| = 1$ if and only if .... (Complete the statement)

(f) Is the conjugacy relation on a group an equivalence relation?

(g) If G is a non-abelian group of order p^3, where p is a prime then $O(Z(G))$ is

(i) either 1 or p

(ii) either p or p^2

(iii) either p^2 or p^3

(iv) either 1 or p^3

(Choose the correct option)

(h) Define normalizer of an element a of a group G.

(i) State Sylow's third theorem.

(j) Let $H = \{(1), (1\,2)\}$. Is H normal in S_3?

2. Answer the following questions : 2×5=10

(a) If $f : G \to \overline{G}$ is an isomorphism from a group G onto a group $\overline{G}$, then prove that f carries the identity of G to the identity of $\overline{G}$.

(b) A group G is abelian if and only if the number of conjugate classes in G is same as the order of G. Is it true? Justify your answer.

Contd.

(c) Express Aut $(U(25))$ in the form $Z_m \oplus Z_n$.

(d) Determine all normal subgroups of D_n of order 2.

(e) Why is any abelian group of order 15 cyclic? Give reason.

3. Answer ***any four*** questions : 5×4=20

(a) Let H be a normal subgroup of a group G and K be any subgroup of G. Prove that HK is a subgroup of G.

(b) Let $Z(G)$ be the centre of a group G. If $G/Z(G)$ is cyclic then prove that G is abelian.

(c) Suppose ϕ is an isomorphism from a group G onto a group $\overline{G}$. If K is a subgroup of G then prove that $\phi(K)=\{\phi(K) \mid k \in K\}$ is subgroup of $\overline{G}$.

(d) Let G and H be finite groups. Prove that $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime.

(e) Prove that a group of order p^2 is abelian, where p a prime.

(f) If m divides the order of a finite abelian group G, then prove that G has a subgroup of order m.

4. Answer the following questions : 10×4=40

(a) Define inner automorphism of a group.

Show that the set $I(G)$ of all inner automorphism of a group G is a subgroup of *Aut (G)*, where *Aut (G)* is the set of all automorphisms of G.

If T_{g_1} and T_{g_2} are *any two* inner auto-morphisms of a group G then show that $T_{g_1} = T_{g_2}$ if and only if $g_1 Z(G) = g_2 Z(G)$ where $Z(G)$ is the centre of G. 1+5+4=10

Or

State and prove Sylow's 2nd theorem. 10

(b) Let G be a group of order p^n, p a prime and n is a positive integer. Show that $O(Z(G)) > 1$ where $Z(G)$ is the centre of G.

Also determine the $O(Z(G))$, where G is a non-abelian group of order p^3. 7+3=10

Contd.

Or

Prove that every p-subgroup of a finite group G is contained in some Sylow p-subgroups of G. 10

(c) Let G be a finite abelian group. Prove that G is isomorphic to the direct product of its Sylow subgroups. 10

Or

If a group G is the internal direct product of a finite number of subgroups $H_1, H_2, \ldots, H_n$ then prove that G is isomorphic to the external direct product of $H_1, H_2, \ldots, H_n$.

(d) Let G be a finite abelian group and $p \mid O(G)$, where p is a prime. Show that there exists an element x in G such that $O(x) = p$. 10

Or

Determine the number of elements of order 5 in $Z_{25} \oplus Z_5$.

Prove that the order of an element in a direct product of a finite number of finite groups is the least common multiple of the orders of the components of the element. In symbols, $|(g_1, g_2, \ldots, g_n)| = l.c.m(|g_1|, |g_2|, \ldots, |g_n|)$.

4+6=10

OPTION–G

(Mathematical Finance)

Paper : MAT–HE–6076

1. Answer the following as directed: 1×10=10

(a) "Forward contracts can be used to hedge foreign currency risk."

*(True **or** False).*

(b) Consider a stock that pays no dividend and is worth Rs. 60. If you can borrow or lend money for 1 year at 5%, what is the 1-year forward price of the stock?

(c) Write the name of the types of options.

(d) What is arbitrage?

(e) Write the full form of OTC.

(f) A combination of a stop order and a limit order is known as ______.

(Fill in the blank)

(g) Write the relation among the value of a swap, the floating-rate bond and the value of the fixed-rate bond.

(h) What is implied volatility?

(i) What is meant by the 'Rho' of a portfolio of options?

(j) One index option contract is on ____ times the index. *(Fill in the blank)*

2. Answer the following: 2×5=10

(a) Explain the difference between hedging and speculation.

(b) Justify that CAPM is a pricing model.

(c) Define the term 'options'.

(d) Explain what you mean by the credit risk and the market risk in a financial contract.

(e) What are the formulas for *u* and *d* in terms of volatility?

3. Answer ***any four*** parts: 5×4=20

(a) A debt of Rs. 55,000 is to be amortized over 9 years at 9% interest p.a. compounded annually. What value of monthly payments will achieve this?

(b) Mr. Bori buys European put option with a strike price of Rs. 100 per share to purchase 100 shares of ABC Company after 4 months. Option price is Rs. 5 per share. If the price of one share is Rs. 98 on expiration date. What will be Mr. Bori's gain/loss if the option is exercised? Should he exercise the option in this case?

(c) Rs. 1,570 is invested at 12% p.a. compound interest. After how many years will the investment be worth Rs. 23,000?

(d) A call option on a non-dividend paying stock has a market price of Rs. 2.50. The stock price is Rs. 15, the exercise price is Rs. 13, the time to maturity is 3 months, and the risk-free interest rate is 5% per annum. What is the implied volatility?

(e) What is meant by the gamma of an option position? What are the risks in the situation where the gamma of a position is highly negative and the delta is zero?

(f) Explain what is meant by

(i) the 3-month LIBOR rate;

(ii) the 3-month OIS rate.

Which is higher? Why?

4. Answer the following questions : 10×4=40

(a) Explain the following terms briefly : 2½×4=10

(i) Zero rates

(ii) Currency scoops

(iii) Put-call party

(iv) Types of traders

Or

Companies *A* and *B* have been offered the following rates per annum on a Rs. 2 lakh 5-year loan :

	Fixed rate	Floating rate
Company *A*	5.0%	LIBOR +0.1%
Company *B*	6.4%	LIBOR +0.6%

Company *A* requires a floating rate loan, Company *B* requires a fixed rate loan. Design a swap that will net a bank, acting as intermediary, 0.1% per annum and that will appear equally attractive to both companies. 10

(b) The spot price of silver is Rs. 25 per gram. The storage costs are Rs. 0.24 per gram per year payable quarterly in advance. Assuring that interest rates are 5% per annum for all maturities, calculate the futures price of silver for delivery in 9 months. 10

Or

An investor receives Rs. 1100 in one year in return for an investment of Rs. 1000 now. Calculate the percentage return per annum with

(i) annual compounding;

(ii) semi-annual compounding;

(iii) monthly compounding;

(iv) continuous compounding.

(c) What are the most important aspects of the design of a new futures contract? Explain how margin accounts protect investors against the possibility of default. 10

Or

A stock index is currently 300, the dividend yield on the index is 3% per annum, and the risk-force interest is 8% per annum. What is a lower bound for the price of a six-month European call option on the index when the strike price is 290?

(d) What does the Black-Scholes-Merton stock option pricing model assume about the probability distribution of the stock price in one year? What does it assume about the probability distribution of the continuously compounded rate of return on the stock during the year? 10

Or

Let $C(k, t)$ be the cost of a cell option on a specified security that has strike price k and expiration time t.
Prove that

(i) for fixed expiration time t, $C(k,t)$ is a convex and nonincreasing function of k; 5

(ii) for $s > 0$, $C(k, t) - C(k+s, t) \leq se^{-rt}$. 5
