Total number of printed pages–16

3 (Sem–6/CBCS) MAT HC 1 (N/O)

2023

MATHEMATICS

(Honours Core)

Paper : MAT–HC–6016

(New Syllabus/Old Syllabus)

Full Marks : 80/60

Time : Three hours

The figures in the margin indicate full marks for the questions.

New Syllabus

Full Marks : 80

(Riemann Integration and Metric Spaces)

1. Answer the following as directed : $1\times10=10$

   (a) Define the discrete metric d on a non-empty set X.

Contd.

(b) Let F_1 and F_2 be two subsets of a metric space (X, d). Then

(i) $\overline{F_1 \cup F_2} = \overline{F_1} \cap \overline{F_2}$

(ii) $\overline{F_1 \cup F_2} = \overline{F_1} \cup \overline{F_2}$

(iii) $\overline{F_1 \cap F_2} = \overline{F_1} \cap \overline{F_2}$

(iv) $\overline{F_1 \cap F_2} = \overline{F_1} \cup \overline{F_2}$

(Choose the correct option)

(c) Let (X, d) be a metric space and $A \subset X$. Then

(i) $Int\,A$ is the largest open set contained in A.

(ii) $Int\,A$ is the largest open set containing A.

(iii) $Int\,A$ is the intersection of all open sets contained in A.

(iv) $Int\,A = A$

(Choose the correct option)

(d) Let (X, d) be a disconnected metric space.
We have the statements :

I. There exists two non-empty disjoint subsets A and B, both open in X, such that $X = A \cup B$.

II. There exists two non-empty disjoint subsets A and B, both closed in X, such that $X = A \cup B$.

(i) Only I is true

(ii) Only II is true

(iii) Both I and II are true

(iv) None of I and II is true

(Choose the correct option)

(e) Find the limit points of the set of rational numbers Q in the usual metric R_u.

(f) In a metric space, the intersection of infinite number of open sets need not be open. Justify it with an example.

(g) Define a mapping $f : X \to Y$, so that the metric spaces $X = [0, 1]$ and $Y = [0, 2]$ with usual absolute value metric are homeomorphic.

(h) Define Riemann sum of f for the tagged partition (P, t).

(i) State the first fundamental theorem of calculus.

(j) Examine the existence of improper Riemann integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

2. Answer the following questions : 2×5=10

(a) Prove that in a metric space (X, d) every open ball is an open set.

(b) Prove that the function $f : [0,1] \to R$ defined by $f(x) = x^2$ is an uniformly continuous mapping.

(c) Let d_1 and d_2 be two matrices on a non-empty set X. Prove that they are equivalent if there exists a constant K such that

$$\frac{1}{K} d_2(x, y) \leq d_1(x, y) \leq K d_2(x, y)$$

(d) If m is a positive integer, prove that $\overline{|m+1} = m!$

(e) Let $f(x) = x$ on $[0,1]$.

Let $P = \left\{ x_i = \frac{i}{4},\ i = 0, \dots 4 \right\}$

Find $L(f, P)$ and $U(f, P)$.

3. Answer the following questions ***(any four)***: 5×4=20

(a) Let (X, d) be metric space and F be a subset of X. Prove the F is closed in X if and only if F^c is open.

(b) Define diameter of a non-empty bounded subset of a metric space (X, d). If A is a subset of a metric space (X, d), then prove that $d(A) = d(\overline{A})$. 1+4=5

(c) Let (X, d) be a metric space. Then prove that the following statements are equivalent :

(i) (X, d) is disconnected.

(ii) There exists two non-empty disjoint subsets A and B, both open in X, such that $X = A \cup B$.

(d) Let $f, g : [a, b] \to R$ be integrable functions. Then prove that $f+g$ is integrable and

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(e) Discuss the convergence of the integral $\int_1^\infty \frac{1}{x^p}\,dx$ for various values of p.

(f) Consider $f : [0, 1] \to R$ defined by $f(x) = x^2$. Prove that f is integrable.

4. Answer the following questions : 10×4=40

(a) (i) Let X be the set of all bounded sequences of numbers $\{x_i\}_{i \geq 1}$ such that $\sup_i |x_i| < \infty$.

For $x = \{x_i\}_{i \geq 1}$ and $y = \{y_i\}_{i \geq 1}$ in X define $d(x, y) = \sup_i |x_i - y_i|$.

Prove that d is a metric on X. 5

(ii) Prove that a convergent sequence in a metric space is a Cauchy sequence. Is the converse true? Justify with an example. 4+1=5

Or

(a) (i) Show that $d(x, y) = \sqrt{|x - y|}$ defines a metric on the set of reals. 4

(ii) Show that the metric space $X = \mathbb{R}^n$ with the metric given by

$d_p(x, y) = \left(\Sigma |x_i - y_i|^p\right)^{1/p}, \quad p \geq 1$

where $x = (x_1, x_2 \ldots\ldots x_n)$ and $y = (y_1, y_2 \ldots\ldots y_n)$ are in $\mathbb{R}^n$ is a complete metric space. 6

(b) (i) Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \to Y$. If f is continuous on X, prove the following : 3+3=6

(i) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for all subsets of B of Y

(ii) $f(\overline{A}) \subseteq \overline{f(A)}$ for all subsets A of X

(ii) Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \to Y$ be uniformly continuous. Prove that if $\{x_n\}_{n \geq 1}$ is a Cauchy sequence in X, then $\{f(x_n)\}_{n \geq 1}$ is a Cauchy sequence in Y. 4

Or

(b) Define fixed point of a mapping $T: X \to X$. Let $T: X \to X$ be a contraction of the complete metric space (X, d). Prove that T has a unique fixed point. 2+8=10

(c) (i) Prove that if the metric space (X, d) is disconnected, then there exists a continuous mapping of (X, d) onto the discrete two element space (X_0, d_0). 5

(ii) Let (X, d) be a metric space and A^0, B^0 are interiors of the subsets A and B respectively. Prove that

$$(A \cap B)^0 = A^0 \cap B^0;$$
$$(A \cup B)^0 \supseteq A^0 \cup B^0.$$
5

Or

(c) (i) When is a non-empty subset Y of a metric space (X, d) said to be connected? Let (X, d_X) be a connected metric space and $f:(X, d_X) \to (Y, d_Y)$ be a continuous mapping. Prove that the space $f(X)$ with the metric induced from Y is connected. 5

(ii) Let (X, d) be a metric space and $Y \subseteq X$. If X is separable then prove that Y with the induced metric is also separable. 5

(d) (i) If f is Riemann integrable on $[a, b]$ then prove that it is bounded on $[a, b]$. 5

(ii) When is an improper Riemann integral said to exist?
Show that the improper integral of $f(x) = |x|^{-1/2}$ exists on $[-1, 1]$ and its value is 4. 1+4=5

Or

(d) (i) Let $f: [a, b] \to R$ be integrable. Then prove that the indefinite integral $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$.

Further prove that if f is continuous at $x \in [a, b]$, then F is differentiable at x and $F'(x) = f(x)$. 3+3=6

(ii) Evaluate

$$\lim_{x \to \infty} \frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{\sqrt{n^3}} = \frac{2}{3}$$
4

Old Syllabus

Full Marks : 60

(Complex Analysis)

1. Answer the following as directed : 1×7=7

(a) Any complex number $z=(x, y)$ can be written as

(i) $z=(0, x)+(1,0)(0, y)$

(ii) $z=(x, 0)+(0,1)(y, 0)$

(iii) $z=(x, 0)+(0,1)(0, y)$

(iv) $z=(0, x)+(1,0)(y, 0)$

(Choose the correct option)

(b) Write the function $f(z)=z^2+z+1$ in the form $f(z)=u(x, y)+iv(x, y)$.

(c) The value of $\lim\limits_{z \to \infty} \frac{2z+i}{z+1}$ is

(i) ∞

(ii) 0

(iii) 2

(iv) i

(Choose the correct option)

(d) Determine the singular points of the function

$$f(z)=\frac{z^2+1}{(z+2)(z^2+2z+2)}$$

(e) Define an analytic function of the complex variable z.

(f) $e^{i(2n+1)\pi}$ is equal to

(i) 1

(ii) −1

(iii) 0

(iv) 2

(Choose the correct option)

(g) $Log(-1)$ is equal to

(i) $\frac{\pi}{2}i$

(ii) πi

(iii) $-\frac{\pi}{2}i$

(iv) $-\pi i$

(Choose the correct option)

2. Answer the following questions : 2×4=8

(a) Show that $\lim_{z \to \infty} \frac{1+z^2}{z-1} = \infty$

(b) If $f(z) = e^x \cdot e^{iy} = e^z$ where $z = x + iy$, show that $f'(z) = e^x \cos y + i e^x \sin y$.

(c) Show that $\int_C f(z)\, dz = 0$ when the contour C is the unit circle $|z| = 1$ in either direction and $f(z) = \frac{z^2}{z-3}$.

(d) Show that the sequence $z_n = \frac{1}{n^3} + i$ $(n = 1, 2, 3, \ldots)$ converges to i.

3. Answer **any three** questions from the following : 5×3=15

(a) If z_1 and z_2 are complex numbers then show that

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

(b) Suppose a function $f(z)$ be analytic throughout a given domain D. If $|f(z)|$ is constant throughout D, then prove that $f(z)$ is constant in D.

(c) Show that the derivative of the real valued function $f(z) = |z|^2$ exists only at $z = 0$.

(d) If a function f is analytic at a given point, then prove that its derivatives of all orders are analytic there too.

(e) State Cauchy integral formula. Apply it to find $\int_C \frac{f(z)}{z+i}\, dz$ where $f(z) = \frac{z}{9-z^2}$ and C is the positively oriented circle $|z| = 2$.

4. Answer **either** (a) and (b) **or** (c) of the following questions : 10

(a) (i) Show that if $f(z) = \frac{i\bar{z}}{2}$ in the open disk $|z| < 1$, then

$$\lim_{z \to 1} f(z) = \frac{i}{2}$$ 3

Contd.

(ii) Show that the function

$f(z) = e^{-y}\sin x - ie^{-y}\cos x$ is entire. 3

(b) If a function $f(z)$ is continuous and nonzero at a point z_0, then prove that $f(z) \neq 0$ throughout some neighbourhood of that point. 4

Or

(c) Let the function

$f(z) = u(x, y) + iv(x, y)$ be defined throughout some ε neighbourhood of a point $z_0 = x_0 + iy_0$, and suppose that

(i) the first order partial derivatives of the functions u and v with respect to x and y exist everywhere in the neighbourhood;

(ii) those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ at (x_0, y_0).

Prove that $f'(z)$ exists and

$f'(z_0) = u_x + iv_x$ where the right hand side is to be evaluated at (x_0, y_0). 10

5. Answer **either** (a) and (b) **or** (c) and (d) of the following questions : 10

(a) Find the value of $\int_C \bar{z}\, dz$ where C is the right-hand half $z = 2e^{i\theta}$ $\left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)$ of the circle $|z| = 2$ from $z = -2i$ to $z = 2i$. 5

(b) Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the 1st quadrant. Show that

$$\left| \int_C \frac{z+4}{z^3 - 1} dz \right| \leq \frac{6\pi}{7}$$ 5

Or

(c) State Liouville's theorem. 1

(d) Prove that any polynomial

$p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n \quad (a_n \neq 0)$

of degree $n\ (n \geq 1)$ has at least one zero. 9

Contd.

5. Answer **either** (a) and (b) **or** (c) and (d) of the following questions : 10

(a) Suppose that $z_n = x_n + iy_n$ $(n = 1, 2, 3 \ldots)$ and $S = X + iY$. Prove that

$$\sum_{n=1}^{\infty} z_n = S \text{ if and only if}$$

$$\sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y.$$ 5

(b) Find the Maclaurin series for the entire function $f(z) = \sin z$. 5

Or

(c) Define absolutely convergent series. Prove that the absolute convergence of a series of complex numbers implies the convergence of the series. 1+3=4

(d) Find the Maclaurin series for the entire function $f(z) = \cos z$. 6

2500

3 (Sem–6/CBCS) MAT HC 2

2023

MATHEMATICS

(Honours Core)

Paper : MAT–HC–6026

(Partial Differential Equation)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following : 1×7=7

   (i) The first order, quasi linear and linear partial differential equation are solved by using

      (a) Lagrange's method

      (b) Charpit's method

(c) Jacobi method

(d) None of the above

(Choose the correct answer)

(ii) The partial differential equation

$$x\left(\frac{\partial^2 z}{\partial x^2}\right)+\frac{\partial^2 z}{\partial y^2}=x^2$$ is classified as

(a) Parabolic, $x = 0$

(b) Elliptic, $x > 0$

(c) Hyperbolic, $x < 0$

(d) All of the above

(Choose the correct answer)

(iii) What are the order and degree of

$$\frac{\partial^2 z}{\partial x^2}=\sqrt{1+\frac{\partial z}{\partial y}}\ ?$$

(iv) What type of partial differential equation is readily solved by Charpit's method ?

(v) The equation $p^2+q^2=1$ is

(a) linear

(b) semi linear

(c) quasi linear

(d) Non-linear

(Choose the correct answer)

(vi) The solution which has number of arbitrary constants equal to number of independent variables is

(a) general integral

(b) complete integral

(c) particular integral

(d) singular integral

(Choose the correct answer)

(vii) Write down the form obtained of the PDE, in a function $X(x, y)$ and two variables x, y after seperation of variables is applied.

2. Answer in short : 2×4=8

(i) Write down the construction of a first order partial differential equation.

(ii) Define partial differential equation. Give *one* example.

(iii) Eliminate arbitrary constants from $z = Ae^{pt}\sin px$ to form a partial differential equation.

(iv) Determine whether the given equation is parabolic, elliptic or hyperbolic

$$y^2\frac{\partial^2 z}{\partial x^2} - x^2\frac{\partial^2 z}{\partial y^2} = 0$$

3. Answer **any three** : 5×3=15

(i) Eliminate the arbitrary function f from the equation

$$f\left(x^2+y^2+z^2, z^2-xy\right)=0$$

(ii) Find the general integrals of the linear partial differential equations

$$z(xp-yq) = y^2 - x^2$$

(iii) Find the equation of the integral surface of the differential equation $2y(z-3)p + (2x-z)q = y(2x-3)$ which passes through the circle $z=0$, $x^2+y^2=2x$.

(iv) Reduce to canonical form and find the general solution of $u_x + u_y = u$.

(v) Apply the method of seperation of variables $u(x, y) = f(x)\,g(y)$ to solve the equation $y^2u_x^2 + x^2u_y^2 = (xyu)^2$.

4. Answer the following questions : 10×3=30

(a) Find a complete integral of $\left(p^2+q^2\right)y = qz$ by Charpit's method.

Or

Apply the method of seperation of variables $u(x, y) = f(x)\,g(y)$ to solve the equation $u_x + 2u_y = 0$, $u(0, y) = 3e^{-2y}$.

(b) Solve $p_3 x_3 (p_1 + p_2) + x_1 + x_2 = 0$ by Jacobi method.

Or

Transform the equation to canonical form $u_{xx} + y^2 u_{yy} = y$.

(c) Obtain the general solution of the equation

$$x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy} + xy\,u_x + y^2 u_y = 0$$

Or

Solve the following :

(i) $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

(ii) $(x^2 - y^2 - z^2)p + 2xyq = 2xz$

2500