NEAR-RINGS AND NEAR-RING GROUPS WITH CHAIN CONDITIONS : SOME SPECIAL TYPES

THESIS
SUBMITTED TO GAUHATI UNIVERSITY
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DOCTOR OF PHILOSOPHY IN THE
FACULTY OF SCIENCE

By
MD. ABUL MASUM, M.Sc., M. Phil
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Submitted on
25/4/95

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Date: 25/4/95

(A. MASUM)

C E R T I F I C A T E

This is to certify that the thesis entitled "Near-rings and near-ring groups with chain conditions: some special types" is the outcome of study and investigations carried out by Md. Abul Masum. It has been done under my supervision and guidance and neither the thesis nor any part thereof has been submitted in this or any other university for a research degree.

Md. Abul Masum fulfils the requirements of the regulations relating to the nature and prescribed period of research work for the award of the degree of Doctor of Philosophy of the Gauhati University.

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INTRODUCTION

An algebraic system with binary operations + (addition) and . (multiplication) satisfying all the ring axioms except possibly one of the distributive laws and commutativity of addition, is a near-ring. The collection of set mappings $f : G \to G$ on a group $(G, +)$ [not necessarily abelian] together with the operations of pointwise addition and composition of mappings, furnishes the most fruitful example of a near-ring.

Extensive research works are being carried out on near-rings and near-ring groups in which structure theory is of another area of importance. Here we devote ourselves to obtain some special types of structure theorems on near-rings satisfying different chain conditions. St. Ligh, Beidleman, Oswald, Goyal and others have done considerable work on various aspects of near-rings with chain conditions on annihilators. Oswald and Goyal's work on near-rings with chain conditions on annihilators are elegant and noticeable. Chain condition on annihilators has its own beauty. Because of non ring character of a near-ring, independent observation in case of chain conditions on right and left annihilators is of considerable importance.

In this dissertation, we confine ourselves to near-rings and near-ring groups with various types of chain conditions on annihilators together with finite spanning dimension (fsd) and finite

Goldie dimension (fgd) characters. We extend the notion of fsd in modules over a ring due to Fleury [18] to near-ring groups. It is to be noted that the fsd defined here is different from that in Satyanarayana [44], [46] and Co-workers [47] (Ref. Page 748 [[15]]) and the first one will be denoted by fsd-1. We obtain a supplement primary (s-p) decomposition of the zero (0) of an fsd-1 near-ring group.

We observe visible differences between a right Goldie near-ring and a left Goldie near-ring as defined by Tamuli and Chowdhury [52] and here, in both cases, ascending chain condition on annihilators is an essential condition to be satisfied. So, it is of obvious importance to study right Goldie near-rings and left Goldie near-rings separately. Here, we have obtained Goldie theorem analogue for a left Goldie near-ring. In some special cases, results what may be termed as a partial converse of the above results, have been obtained. In [10] , Chowdhury has already established some results on near-rings of right quotients of a right Goldie near-ring and on some radical characters. Here, in the other chapters, we deal with right Goldie near-rings extending some results of A.Oswald.

Important and interesting investigation is carried out in this dissertation on near-rings with fgd satisfying the acc on annihilators with parts having minimum conditions. Near-rings with acc on near-ring subgroups always satisfy the acc on annihilators and are also with fgd. But the converse is not true, for there exists examples of near-rings (rings) satisfying these two conditions but the acc on its ideals. Yet such near-rings may contain parts satisfying the acc(dcc) on its invariant subnear-rings (ideals). Such type of subalgebraic structures seem to be worthnoticing. Many a good problems would possibly draw the attention of the researchers working in this area. Outcome of our work along the line mentioned above has been described in five chapters accordingly.

The first chapter is nothing but our endeavour to introduce near-rings and near-ring groups through the preliminary definitions and properties to make our discussion a self contained one.

The second chapter is the outcome of our paper [[15]] published in IJPAM. Here we find the propogation of chain conditions from a near-ring group E to its attached near-ring N - a generalisation of the theorem of Oswald [41] . If E is with fsd-1, then E and $E/_H$ satisfy the chain conditions on their substructures where H is a non-zero supplement in E. Results along the line of P.Fleury [18] and B.Satyanarayana [44] lead us to s-p decomposition of (O) in E as $\bigcap_{i=1}^{t} E_i = (0)$ where $E_i^{!s}$ are supplements in E and the collection $\mathcal{A}(E) = \bigcup_{i=1}^{t} \mathcal{A}(E/_{E_i})$. Also we prove here that for any x belonging to each of the associated strictly primes of E, a power of it belongs to the annihilator of E in some special cases.

Non singularity of a left Goldie near-ring leads us to prove the result (theorem 7) of Oswald [41] independently ([[13]]) in

Chapter III and plays a key role in what we have discussed later in this chapter such as : When N is a strongly semiprime strictly left Goldie (right) near-ring, the family of maximal left annihilators of invariant subnear-rings is a finite one and with zero intersection. If N is a distributively generated near-ring (d.g.nr.) with distributively generated left annihilators then in some special cases the complete near-ring Q(N) of left quotients of N coincides with the classical near-ring Q. Moreover, Q has no nilpotent left Q-subgroup and it satisfies the dcc on its left Q-subgroups. Conversely, if Q is a strictly left Goldie near-ring so is also N and if N is a d.g.nr. and Q satisfies the the dcc on its left Q-subgroups then N has no non-zero nilpotent left N-subgroup.

We have proved some special properties of a strongly semiprime strictly left Goldie near-ring in the fourth chapter which is the outcome of our papers [[14]] and [[37]] . A subnear-ring of a Goldie near-ring need not be Goldie. But some properties of Goldie near-ring (existence of classical near-ring in particular) are inherited by subnear-ring (without being Goldie) when the parent near-ring is radical over it.

Moreover, a near-ring with acc on annihilators having no infinite direct sum of ideals (near-ring subgroups) need not satisfy the acc(dcc) on its subalgebraic structures e.g. commutative integral domain like $Z\left[X_i \mid i = 1,2,\ldots;\ X_iX_j = X_j X_i\right]$. But it may contain some parts satisfying the acc(dcc) on the same.

A left singular subset modulo maximal annihilator of a left Goldie near-ring leads us in some special cases to the cyclic structure of an ideal I satisfying the dcc on its right N-subgroup. A sufficient condition is established on an ideal (which is minimal as an invariant subnear-ring with dcc on its right N-subgroups) to be a near-ring group over a near-ring with dcc on its near-ring subgroups which is an extension of an epimorphic image of N. If N is a strongly semiprime strictly left Goldie ndar-ring where every weakly essential left N-subgroup is essential then in case of a countable ideal I with dcc on its N-subgroups, the N-group $N/_{l(I)}$ also inherits the same character as I.

Structural difference of left and right annihilators in a near-ring N envisaged us to study what is termed as a right Goldie near-ring in the fifth chapter. We have shown that every ideal I of a weakly regular d.g.nr. with acc on right annihilators possesses an identity which is central idempotent. This leads us to the structure theorem such as : Weakly regular d.g.nr. with acc on right annihilators is a direct sum of ideals which are weakly regular simple d.g.nrs. with identities. Moreover, in case of a strongly prime right Goldie near-ring N with d.g.left annihilators, we have got a quasi near-domain $M/_{(J\cap M)}$ for a left annihilator M (= l(J)) of maximal right annihilator J in N.

ABBREVIATIONS AND SYMBOLS

acc	- ascending chain condition
dcc	- descending chain condition
d.g.	- distributively generated
d.g.nr.	- distributively generated near-ring
fgd	- finite Goldie dimension
fsd	- finite spanning dimension
w.r.t.	- with respect to
CLMP	- common left multiple property
l.e.dcc	- left essential descending chain condition
IJPAM	- Indian Journal of Pure and Applied Mathematics.
N	- near-ring
$\overline{N}$	- factor near-ring
E or ${}_NE$	- near-ring group (N-group)
$\overline{E}$	- factor N-group
Z^+	- set of positive integers.
Z_p	- group of integers under addition modulo p (for $p \in Z^+$)
$\oplus$	- direct sum
$\sum_{fin}$	-finite sum
$\trianglelefteq$	- ideal
$\wedge$	- omission of the symbol underneath it.
$\langle a \rangle$	- a group generated by a
$\langle S \rangle$	- a group generated by a set S

$_I\langle x \rangle$	- a left I-subgroup of I generated by $x(\in I)$.
$l_I(S)$	$= \{x \in I \mid xS = (0)\}$ where $S \subseteq N$ and I is a subnear-ring of N.
$l(A)$	$= l_N(A)$
$r_I(S)$	$= \{x \in I \mid Sx = (0)\}$ where $S \subseteq N$ and I is a subnear-ring of N.
$r(A)$	$= r_N(A)$
$Ann(M)$	$= \{n \in N \mid nm = 0,$ for all $m \in M\}$ where $M \subseteq E$
$r_E(A)$	$= \{x \in E \mid ax = 0.$ for all $a \in A\}$ where $A \subseteq N$
$(I; x)$	$= \{n \in N \mid nx \in I\}$
$\subseteq_e$	- essential
$\subset_s$	- small
$\leq_c$	- closed
$\cong$	- isomorphic
$h\|_M$	- restriction of h to M.

$Hom_1(M,E) = \{f : M \to E \mid$ N-homomorphic image $f(M) \trianglelefteq E\}$

$Sd_1(L)$ - spanning dimension of L

$Z_1(E) = \{e \in E \mid Le = (0)$ for some essential N-subgroup L of $_NN\}$

$Z_1(N) = \{x \in N \mid Ax = (0)$ for some essential left N-subgroup A of N $\}$

$\mathcal{A}(E) = \{P \mid P = Ann(M)$ for some prime N-subgroup M of E $\}$

$\mathcal{P}$	$= \{ P \mid P$ is a maximal left annihilator of non-zero left N-subset of N $\}$
Γ	$= \{ P \mid P$ is a maximal left annihilator of non-zero invariant subnear-ring of N $\}$
Q or C_{cl}	- classical near-ring of left quotients of N
Q(N)	- complete near-ring of left quotients of N
[[ ]]	- to refer the paper of the author.

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CHAPTER I

Preliminaries

In this introductory chapter, we attempt to cover enough of the fundamental concepts of near-rings and near-ring groups to make our investigation self contained. This chapter has been divided into four sections where the first one contains the (basic) fundamental definitions. The second section contains different substructures of near-rings and near-ring groups and their relationship together with some results on them. The notion of left (right) annihilators of subsets of a near-ring and a near-ring group have been included in the third section. Various preliminary results on annihilators have also been discussed. The last section contains, for the sake of completeness, definitions of some key terms such as chain condition, direct sum, homomorphisms, some well known lemmas together with the fundamental results.

1.1. Near-rings and Near-ring groups.

1.1.1. Definitions : A triple $(N, +, .)$, where N is a non empty set, $+$ and $.$ are two operations in N, is called a right near-ring if

(i) (N,+) is a group (not necessarily Abelian)

(ii) (N, .) is a semigroup and

(iii) (a+b).c = a.c + b.c, for a,b,c ε N. [If (iii) is replaced by (iv) a.(b+c) = a.b + a.c then the corresponding triple is called a left near-ring.]

Unless otherwise specified, ab will mean a.b for a,b ε N.

Obviously, every ring is a left as well as a right near-ring. So, a near-ring can be called as a generalised ring. Another natural example of a right near-ring is as follows :

1.1.2. Example : The set M(G) of all mappings of a (additively written) group G into itself with addition and multiplication defined by

$(f+g)(a) = f(a) + g(a)$ and

$(fg)(a) = f(g(a))$ for all a ε G and f,g ε M(G) forms a right near-ring.

We confine our discussion on right near-ring only. From now onwards, unless otherwise specified, a near-ring (nr.) will mean a right near-ring.

1.1.3. Definition : The additive identity of the group (N,+) of a near-ring N is called a zero element and it is denoted by O.

1.1.4. Property : For each a in a near-ring N, $0.a = 0$ (zero element of N).

1.1.5. Definition : A near-ring N is called zero symmetric if $a.0 = 0$, for all $a \in N$.

1.1.6. Definition : If the semigroup (N, .) of a near-ring N possesses an element 1 such that $a.1 = a = 1.a$, for all $a \in N$ then 1 is called the identity or unity of N.

Throughout our discussion, N will stand for a zero symmetric near-ring with unity 1.

1.1.7. Definitions : An element $x \in N$ is called an idempotent if $x^2 = x$. Moreover, an idempotent x is called central if $ax = xa$ for all $a \in N$. It is clear that the identity (if exists) of N is always a central idempotent.

An element $x \in N$ is called a nilpotent if for some $t \in Z^+$, $x^t = 0$.

An element $d \in N$ is called distributive if $d(a+b) = da + db$, for all $a,b \in N$.

1.1.8. Properties : (i) If $a,b \in N$, then $(-a)b = -ab$. But $a(-b) = -ab$ is not always true.

(ii) If d is a distributive element of N then $d.0 = 0$, $d(-a) = -da$ and $(-d)(-a) = da$, for all $a \in N$.

1.1.9. Definition : An element $x \in N$ is said to be regular if there exists an element $y \in N$ such that $xyx = x$.

1.1.10. Definition : If N is a near-ring then the group $(E,+)$ is called an N-group (near-ring group) ${}_NE$ when there exists a map $N \times E \to E$, $(n,e) \to ne$ such that

(i) $(n_1 + n_2)e = n_1e + n_2e$

(ii) $(n_1n_2)e = n_1(n_2e)$

(iii) $1.e = e$, for all $n_1, n_2 \in N$, $e \in E$.

In what follows, E will stand for the near-ring group ${}_NE$.

Clearly near-ring N can always be considered as an N-group. We shall write ${}_NN$ to denote N as an N-group.

1.1.11. Example (Ex. 1.18(c) [42]) : Let G be an additive group and $M(G)$ be a near-ring defined in 1.1.2., then G is a $M(G)$ - group when $M(G) \times G \to G$ such that $(f,x) \to f(x)$, for $x \in G$, $f \in M(G)$.

1.1.12. Example : Every left module M over a ring R is an R-group over the near-ring R.

1.1.13. Properties : If E is an N-group then

(i) $0.e = 0$ (the first 0 is the zero element of N and the second 0 is the zero element of E).

(ii) $(-n)e = -ne$ and

(iii) $(n-n_1)e = ne - n_1e$, for all $e \in E$; $n,n_1 \in N$.

1.2. Substructures :

If A and B are two subsets of a near-ring N then we consider the set $AB = \{ab \mid a \in A, b \in B\}$.

Now, we find some substructures of a near-ring given below:

1.2.1. Example (E(2), Page 339-340, [42]):

$N = \{0,a,b,c\}$ is a near-ring under the operations defined by the following tables.

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

(i)

.	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
c	0	a	c	c

(ii)

Table : 1.1

Here we note that if $A = \{0,a\}$, $B = \{0,a,b\}$ and $C = \{0,a,c\}$ are subsets of N then $BN \subseteq B$, $CN \subseteq C$ whereas $NA \subseteq A$ and $AN \subseteq A$. Thus, we define the following

1.2.2. Definitions: A non-empty subset S of a near-ring N is said to be

(i) a right N-subset of N if $SN \subseteq S$

(ii) a <u>left N-subset</u> of N if $NS \subseteq S$ and

(iii) an <u>invariant subset</u> of N if $NS \subseteq S$, $SN \subseteq S$.

It is clear that an invariant subset of a near-ring N is a left as well as right N-subset of N. Moreover, every left(right) N-subset contains the zero element of N.

We now define regularity of a near-ring in a general setting in relation to what is available.

1.2.3. <u>Example</u> (E(14), page 339-340[42]) :

$N = \{0,a,b,c\}$ is a near-ring under addition [defined in table 1.1.(i)] and multiplication defined by the following table

.	0	a	b	c
0	0	0	0	0
a	0	a	0	c
b	0	0	0	0
c	0	a	0	c

Table : 1.2

Here $A = \{0,a,c\}$ is an invariant subset of N. For each $x \in A$, we have an element $y \in A$ such that $x = xyx$ for $a = aca$, $c = cac$.

Now, we define the following

1.2.4. <u>Definitions</u> : An invariant subset A of a near-ring N is called <u>regular</u> if each $x \in A$ is regular .

A near-ring N is called <u>regular</u> if it is regular as an invariant subset.

1.2.5. <u>Definitions</u> : A right (left) N-subset S of a near-ring N is called <u>nil</u> if each element of S is nilpotent.

A subset S of a near-ring N is called a <u>nilpotent subset</u> if there exists a +ve integer t such that $S^t = (0)$ [i.e., $a_1 a_2 \ldots a_t = 0$ for $a_i \in S$, $i = 1, 2, \ldots, t$.]

1.2.6. <u>Lemma</u> : Intersection of two left (right) N-subsets of a near-ring N is again a left(right) N-subset of N.

<u>Proof</u> : Let A and B be two left N-subsets of N. Then $x \in A \cap B \Rightarrow x \in A$ and $x \in B$. Thus for $n \in N$, $nx \in A$ and $nx \in B$ as A, B are left N-subsets of N.

Therefore, $nx \in A \cap B$. This gives, $A \cap B$ is a left N-subset of N. //

1.2.7. <u>Extension</u> : Intersection of any number of left (right) N-subsets of a near-ring N is also a left(right) N-subset of N.

1.2.8. <u>Corollary</u> : Intersection of any number of invariant subsets of a near-ring N is also an invariant subset of N.

1.2.9. <u>Lemma</u> : If A is a left N-subset and B is a right N-subset of a near-ring N then AB is an invariant subset of N.

Proof : Since $N(AB) = (NA)B \subseteq AB$ as A is a left N-subset of N and $(AB)N = A(BN) \subseteq AB$ as B is a right N-subset of N, so AB is an invariant subset of N. //

1.2.10. Lemma : If A is a left (right) N-subset of a near-ring N then for all $t \in Z^+$, A^t is a left (right) N-subset of N.

Proof : If $t = 1$ then it is obvious. If $t = 2$ then $NA^2 = (NA)A \subseteq AA$ as A is a left N-subset of N. Thus, $NA^2 \subseteq A^2$. So A^2 is a left N-subset of N.

Now, let A^{r-1} be a left N-subset of N for any $r \in Z^+$ then $NA^{r-1} \subseteq A^{r-1}$.

Therefore, $NA^r = (NA^{r-1})A \subseteq A^{r-1}A = A^r$.

Hence A^r is also a left N-subset of N if A^{r-1} is a left N-subset of N. Therefore, by induction, A^t is a left N-subset of N for any $t \in Z^+$. //

1.2.11 Lemma : (5.1.4, Chowdhury [7]): The sum $A + B = \{a+b \mid a \in A, b \in B\}$ of two right N-subsets A and B of a near-ring N is also a right N-subset of N.

Now, we observe some other substructures of a near-ring given below.

1.2.12. Example (H(2), page 341-342 [42]):

N = {0,a,b,c,x,y } is a near-ring under the operations defined by the following tables.

+	0	a	b	c	x	y
0	0	a	b	c	x	y
a	a	0	y	x	c	b
b	b	x	0	y	a	c
c	c	y	x	0	b	a
x	x	b	c	a	y	0
y	y	c	a	b	0	x

(i)

.	0	a	b	c	x	y
0	0	0	0	0	0	0
a	0	0	a	a	a	a
b	0	0	b	c	c	b
c	0	0	c	b	b	c
x	0	0	x	y	y	x
y	0	0	y	x	x	y

(ii)

Table : 1.3

We see that if $X = \{0,x,y\}$ then X is a sub-group of the near-ring N which satisfies the following : (i) $XN \subseteq X$, (ii) $N X \not\subseteq X$, (iii) $X.X \subseteq X$. Thus, we are in a position to define the following.

1.2.13. Definitions : If N is a near-ring and H is a subgroup of (N,+) then H is called

(i) a left N-subgroup of N if $NH \subseteq H$,

(ii) a right N-subgroup of N if $HN \subseteq H$

(iii) a subnear-ring of N if $H.H \subseteq H$ and

(iv) an <u>invariant subnear-ring</u> of N if $NH \subseteq H$ and $HN \subseteq H$.

A subgroup M of an N-group E over the near-ring N is called an <u>N-subgroup of E</u> if $NM \subseteq M$.

1.2.14. <u>Note</u> : A left N-subgroup H of N is an N-subgroup of N^N and conversely.

It is seen in 1.2.12 that X is a right N-subgroup as well as a subnear-ring of N but it is not an invariant subnear-ring.

1.2.15. <u>Definitions</u> : A subnear-ring A of N is said to be <u>distributively generated (d.g.)</u> if there is a set D of distributive elements of N such that $A = \cap \{S \mid S \supseteq D, (S,+) \text{ is a subgroup of } (N,+)\}$.

If N, as a subgroup of $(N,+)$, is generated by the set D of all distributive elements of N then we say that <u>N is d.g.</u> and we write $N = \langle D \rangle$.

It is clear that an arbitrary element of $N = \langle D \rangle$ is of the type $\sum_{fin} \pm d_i,\ d_i \in D$.

1.2.16. <u>Lemma</u> : If A is a left N-subgroup of a near-ring N and $x \in N$ then Ax is a left N-subgroup of N.

<u>Proof</u> : Since A is a left N-subgroup of N,

$N(Ax) = (NA)x \subseteq Ax$. Hence Ax is a left N-subset of N.

Also, if ax, bx are any two elements of Ax for a,b ϵ A then ax − bx = (a−b)x ϵ Ax as a−b ϵ A.

Hence Ax is a left N-subgroup of N. //

1.2.17. Corollary : Nx is always a left N-subgroup of N.

1.2.18. Lemma : If A is a right N-subgroup of a near-ring N and x ϵ N then xA is a right N-subset of N.

Proof : Since A is a right N-subgroup of N, $(xA)\,N = x(AN) \subseteq xA$. Hence xA is a right N-subset of N. //

1.2.19. Corollary : For x ϵ N, xN is a right N-subset of N.

1.2.20. Lemma : Intersection of two left (right) N-subgroups of a near-ring N is a left(right) N-subgroup of N.

Proof : Let A,B be two left N-subgroups of N and so [by 1.2.6] $A \cap B$ is a left N-subset of N. Since A, B are subgroups of (N,+) and so $A \cap B$ is also a subgroup of (N,+). Thus $A \cap B$ is a left N-subgroup of N. //

Combining the above result for left as well as right N-subgroups, we have

1.2.21. Lemma : Intersection of two invariant subnear-rings of a near-ring N is an invariant subnear-ring of N.

1.2.22. <u>Lemma</u> : If I is a left N-subgroup of a near-ring N and $x \in N$ then $(I : x) = \{n \in N \mid nx \in I\}$ is a left N-subgroup of N.

<u>Proof</u> : Let $a,b \in (I : x)$ then $ax, bx \in I$.

Therefore, $(a-b)x = ax - bx \in I$

$\Rightarrow$ $a - b \in (I : x)$

Thus, $(I : x)$ is a subgroup of N.

Also, if $y \in (I : x)$ then $yx \in I$.

Therefore, for any $n \in N$, we have

$n(yx) \in NI \subseteq I$

$\Rightarrow$ $(ny)x \in I$

$\Rightarrow$ $ny \in (I : x)$

Hence $(I : x)$ is a left N-subgroup of N. //

It is observed that the following four results are similar to that of a module over a ring and so their proofs are omitted.

1.2.23. <u>Lemma</u> : Intersection of two N-subgroups of $_NE$ is an N-subgroup of $_NE$.

1.2.24. <u>Lemma</u> : If M is an N-subgroup of $_NE$ and S is an N-subgroup of $_NM$ then S is an N-subgroup of $_NE$.

1.2.25. <u>Lemma</u> : If S,M are N-subgroups of $_NE$ such that $S \subseteq M$ then S is an N-subgroup of $_NM$.

1.2.26. Lemma : If A is an N-subgroup of ${}_N E$ and I is an N-subgroup of ${}_N N$ then Ia is an N-subgroup of A for $a \in A$.

Hence Na is also an N-subgroup of A.

1.2.27. Definitions : Let I be an additive normal subgroup of a near-ring N then I is called

(i) a right ideal of N if $in \in I$, for all $i \in I$, $n \in N$.

(ii) a left ideal of N if $n_1(i+n_2)-n_1n_2 \in I$, for all $i \in I$, $n_1, n_2 \in N$.

(iii) an ideal of N if I is a right as well as a left ideal of N.

The condition (ii) for a left ideal is equivalent to $n_1(n_2 + i) - n_1n_2 \in I$, for all $i \in I$ and $n_1, n_2 \in N$ as I is an additive normal subgroup of N.

We write $I \trianglelefteq N$ when I is an ideal of N. All ideals of N except $\{0\}$ and N are called proper ideals of N.

If near-ring N does not contain any proper ideal of it then N is called a simple near-ring.

It is clear that an ideal is itself a near-ring and so every ideal of a near-ring N is a subnear-ring of N.

1.2.28. Lemma (2.15, page 46[42]) : If A is a left N-subgroup of a near-ring N and B is a left ideal of N then $A+B = B+A$ is a left N-subgroup of N.

Proof : Clearly A+B is a subgroup of N. Now, if $n \in N$, $a \in A$, $b \in B$ then

$n(a+b) = n(a+b) - na + na \in B + A$ as B is a left ideal of N and A is a left N-subgroup of N.

Therefore, $n(a+b) \in B+A = A+B$ as B is a normal subgroup of N.

This gives, $N(A+B) \subseteq A+B$. So A+B is a left N-subgroup of N. //

1.2.29. Lemma : If I is a left ideal of N then $(I ; x) = \{ n \in N \mid nx \in I \}$ is a left ideal of N for $x \in N$.

Proof : Since left ideal is a left N-subgroup, [by 1.2.22] $(I ; x)$ is a subgroup of N.

Now, if $n \in N$, $y \in (I ; x)$ then

$(n+y-n)\, x = nx + yx - nx$

$= n_1 + i - n_1$, where $n_1 = nx \in N$ and $i = yx \in I$

$\Rightarrow (n+y-n)\, x = n_1+i-n_1 \in I$ as I is a left ideal of N.

$\Rightarrow n + y - n \in (I ; x)$

Hence $(I ; x)$ is a normal subgroup of N.

Again, if $n, n_2 \in N$, $y \in (I ; x)$ then

$[n(y+n_2) - n\, n_2]\, x = [n\, (y+n_2)]\, x - (n\, n_2)\, x.$

$= n(yx + n_2x) - n(n_2x)$

$= n(i+n_3)-nn_3$, where $i = yx \in I$ and $n_3 = n_2x \in N$.

Therefore, $[n(y+n_2) - nn_2]x = n\,(i+n_3)-nn_3 \in I$

$\Rightarrow\; n\,(y + n_2) - nn_2 \in (I\,;\,x)$

Thus $(I\,;\,x)$ is a left ideal of N. //

1.2.30. Definitions : A normal subgroup M of an N-group E is called an ideal of E if, for all $n \in N$, $m \in M$, $e \in E$, $n(e+m) - ne \in M$.

Since M is a normal subgroup of E and so the above condition may be replaced by $n(m+e) - ne \in M$.

Thus the left ideal of a near-ring N coincides with the ideal of ${}_N N$.

Now, we give some useful results related to ideals of an N-group.

1.2.31. Lemma : Intersection of two ideals of an N-group E is again an ideal of E.

1.2.32. Lemma : If M is an ideal of an N-group E and S is an N-subgroup of E such that $M \subseteq S$ then M is an ideal of an N-group S.

1.2.33. Lemma : If H is an N-subgroup of an N-group E and M is an ideal of E then $H \cap M$ is an ideal of N-group H.

Proof : Clearly, $H \cap M$ is a subgroup of H. Since M is a normal subgroup of E, so $H \cap M$ is a normal subgroup of H.

Now, if $n \in N$, $h \in H$ and $x \in H \cap M$ then $nh \in H$ and $n(h+x) \in H$.

Therefore, $n(h + x) - nh \in H$.

So, $H \cap M$ is an ideal of H. //

1.2.34. <u>Lemma</u> : If I and J are ideals of an N-group E then $I + J$ is also an ideal of E.

<u>Proof</u> : Let $i + j$, $i_1 + j_1 \in I + J$ for $i, i_1 \in I$; $j, j_1 \in J$.

Therefore, $(i + j) - (i_1 + j_1)$

$= i + j - j_1 - i_1$

$= i + j_2 - i_1$, where $j_2 = j - j_1 \in J$.

$= i - i_1 + j_3$ [$j_3 \in J$ — a normal subgroup of E.]

$= i_2 + j_3$, [$i_2 = i - i_1 \in I$]

$\Rightarrow (i + j) - (i_1 + j_1) = i_2 + j_3 \in I + J$.

Thus $I + J$ is a subgroup of E.

Also, for $e \in E$, $e + (i + j) - e$

$= (e + i - e) + (e + j - e) \in I + J$ as I, J are ideals of E.

Hence $I + J$ is a normal subgroup of E.

Now, for $n \in N$, we have

$n(e + i + j) - ne$

$= n(e_1 + j) - ne_1 + ne_1 - ne$, [$e_1 = e + i \in E$]

$= j_1 + n(e+i) - ne$, [$j_1 = n(e_1+j) - ne_1 \in J$.]

$= j_1 + i_1$, [$i_1 = n(e+i) - ne \in I$]

$\Rightarrow$ $n(e + i + j) - ne = j_1 + i_1 \varepsilon J + I = I + J$

Therefore, I + J is an ideal of E. //

As a corollary to the above, we get

1.2.35. Corollary : The sum of two left ideals of a near-ring N is also a left ideal of N.

1.2.36. Modular law : If A,B,C are ideals of a near-ring group E such that $B \subseteq A$ then $B + (C \cap A) = (B + C) \cap A$.

1.2.37. Definitions : If I is an ideal of an N-group E then the set $E/_I$ ($= \bar{E}$) of cosets $e + I$ ($= \bar{e}$) of I, $e \varepsilon E$, under the operation + defined by

(i) $(e + I) + (e_1 + I) = (e + e_1) + I$ and

(ii) $n(e+I) = ne + I$, for $n \varepsilon N$, $e, e_1 \varepsilon E$.

forms an N-group and this N-group $E/_I$ is called a factor N-group . We write, $-e + I$ for $-(e + I)$.

If E = N, the corresponding factor N-group $N/_I$ ($=\bar{N}$) becomes a near-ring called a factor near-ring when I is an ideal of N and the condition (ii) above is replaced by

(ii)′ $(n + I)(n_1 + I) = nn_1 + I$ for $n, n_1 \varepsilon N$.

1.2.38. Lemma : If I and M are ideals of an N-group E such that $I \subseteq M$ then $M/_I$ is an ideal of $E/_I$.

Proof : Here I is an ideal of M [by 1.2.32] and thus $M/_I$ exists

Let $m + I$, $m_1 + I \in M/_I$ then

$(m+I) - (m_1 + I) = (m - m_1) + I \in M/_I$

Now, if $e + I \in E/_I$ for $e \in E$ then

$(e + I) + (m + I) - (e + I) = (e + m - e) + I \in M/_I$

Thus, $M/_I$ is a normal subgroup of $E/_I$.

Again, if $n \in N$ then we have

$n [(e+I) + (m+I)] - n (e + I)$

$= n[(e + m) + I] - ne + I$

$= (n(e+m) - ne) + I \in M/_I$

Therefore, M/I is an ideal of $E/_I$. //

1.2.39. Lemma : Let I be an ideal of an N-group E such that $I \subseteq M$ then $M/_I$ is an N-subgroup of an N-group $E/_I$ if and only if M is an N-subgroup of E.

Proof : Let M be an N-subgroup of E then [by 1.2.32] I is an ideal of M. Thus $M/_I$ exists.

If $m + I$, $m_1 + I \in M/_I$ for $m, m_1 \in M$ then we have

$(m + I) - (m_1 + I) = (m - m_1) + I \in M/_I$

Thus, $M/_I$ is a subgroup of $E/_I$.

Now, for $n \varepsilon N$, $n(m + I) = nm + I \varepsilon M/_I$

Hence $M/_I$ is an N-subgroup of $E/_I$.

Conversely, let $M/_I$ be an N-subgroup of $E/_I$. Thus $M \subseteq E$.

If $m + I \varepsilon M/_I$ for $m \varepsilon M$ and $n \varepsilon N$ then

$n(m + I) \varepsilon M/_I$

$\Rightarrow$ $nm + I \varepsilon M/_I$

$\Rightarrow$ $nm \varepsilon M$

Also, $(m + I) - (m_1 + I) \varepsilon M/_I$ for $m, m_1 \varepsilon M$ as $m+I$, $m_1+I \varepsilon M/_I$

Thus, $(m - m_1) + I \varepsilon M/_I$

$\Rightarrow m - m_1 \varepsilon M$.

Therefore, M is an N-subgroup of E. //

1.3. Annihilators :

1.3.1. Definitions : Let $M \subseteq E$ then the annihilator of M[Ann(M)] in the near-ring N is defined as

$Ann(M) = \{n \varepsilon N | nm = 0, \text{ for all } m \varepsilon M\}$

If E=N then Ann(M) is written as $l_N(M)$ or in short $l(M)$ and it is is called a left annihilator of M in N.

If $S \subseteq N$ then $r_E(S) = \{x \varepsilon E \mid sx = 0, \text{ for all } s \varepsilon S\}$

is defined as the <u>annihilator of S in E</u>.

Also, we note that if E = N then $r_E(S)$ is written as $r_N(S)$ or in short r(S) and it is called a <u>right annihilator of S in N</u>.

1.3.2. <u>Definitions</u> : If I is a subnear-ring of N and $S \subseteq N$ then the <u>left annihilator of S in I</u> is defined as $l_I(S)=\{x \in I \mid xS=(0)\}$ and <u>right annihilator of S in I</u> is defined as $r_I(S)=\{x \in I \mid Sx=(0)\}$.

If I = N then $l_I(S) = l_N(S)$ and $r_I(S) = r_N(S)$ are written as l(S) and r(S) respectively.

1.3.3. <u>Lemma</u> : If $M \subseteq E$ then Ann(M) is a left ideal of N.

<u>Proof</u> : Let x,y $\in$ Ann(M) . Thus xm = 0 = ym for all m $\in$ M.

Therefore, xm − ym = 0

$$\Rightarrow (x-y)\, m = 0$$

$$\Rightarrow x - y \in \text{Ann}(M).$$

Again, for n $\in$ N, (n + x − n) m = nm + xm − nm = 0 as xm = 0. Hence n + x − n $\in$ Ann(M) .

So, Ann(M) is a normal subgroup of N.

Also, for $n_1 \in N$, $[n\,(n_1 + x) - nn_1]\, m$

$$= n(n_1m + xm) - (nn_1)\, m$$

$$= n(n_1m) - n(n_1m), \text{ as } xm = 0$$

$$= 0$$

Thus, $n(n_1 + x) - nn_1 \in Ann(M)$

Therefore, Ann(M) is a left ideal of N. //

1.3.4. <u>Corollary</u> : If $E = N$ then $Ann(M) = l(M)$ is a left ideal of N. //

If $a \in N$ then $l(\{a\})$ is written as $l(a)$ and it is a left ideal of N.

1.3.5. <u>Lemma</u> : If M is an N-subgroup of ${}_N E$ then Ann(M) is an ideal of N.

<u>Proof</u> : Ann(M) is a left ideal of N [by 1.3.3].

Now, if $x \in Ann(M)$ and $n \in N$ then

$(xn) M = x(nM) \subseteq xM = (0)$.

$\Rightarrow$ $xn \in Ann(M)$

Hence Ann(M) is a right ideal also and so Ann(M) is an ideal of N. //

1.3.6. <u>Lemma</u> : If A is a left N-subset of N then $l(A)$ is an ideal of N.

<u>Proof</u> : 1.3.4. gives that $l(A)$ is a left ideal of N.

Now, if $x \in l(A)$ and $n \in N$ then

$(xn) A = x (nA) \subseteq xA = (0)$

$\Rightarrow$ $xn \in l(A)$

Hence $l(A)$ is a right ideal of N.

Therefore, $l(A)$ is an ideal of N. //

1.3.7. <u>Corollary</u> : If $A \subseteq N$ then $l(l(A))$ is an ideal of N. //

1.3.8. <u>Lemma</u> : If $A \subseteq N$ then $r(A)$ is a right N-subset of N.

<u>Proof</u> : Let $x \in r(A)$ and $n \in N$ then

$A(xn) = (Ax)n = (0)$

$\Rightarrow$ $xn \in r(A)$, for all $n \in N$.

Therefore, $r(A)$ is a right N-subset of N. //

It is noted that if $a \in N$ then $r(\{a\})$ is written as $r(a)$ and it is a right N-subset of N.

1.3.9. <u>Lemma</u> : If A is a right N-subset of N then $r(A)$ is an invariant subset of N.

<u>Proof</u> : By 1.3.8, we have $r(A)$ as a right N-subset of N.

Now, if $x \in r(A)$ and $n \in N$ then

$A(nx) = (An)x \subseteq Ax = (0)$

$\Rightarrow nx \in r(A)$ for all $n \in N$.

Hence $r(A)$ is a left N-subset of N.

Therefore, $r(A)$ is an invariant subset of N. //

1.3.10. <u>Corollary</u> : If $A \subseteq N$ then $r(r(A))$ is an invariant subset of N.

1.3.11. <u>Lemma</u> : If $S \subseteq N$ and $M \subseteq E$ then

(i) $Sr_E(S) = (0)$ and (ii) $Ann(M)\mathcal{M} = (0)$.

Proof : Easily follows from definition. //

1.3.12. Corollary : If $E = N$ then $Sr(S) = (0)$ and $l(M)\,M = (0)$.

1.3.13. Corollary : If $T \subseteq S \subseteq N$ and $P \subseteq M \subseteq E$ then

$$Tr_E(S) = (0) \quad \text{and} \quad Ann(M)P = (0)$$

Proof : We have

$$Tr_E(S) \subseteq Sr_E(S) = (0)$$

$$\Rightarrow Tr_E(S) = (0)$$

Also, $Ann(M)\,P \subseteq Ann(M)\,M = (0)$

$$\Rightarrow \quad Ann(M)P = (0) \ . \ //$$

1.3.14. Corollary : If $E = N$ in 1.3.13 then

$$Tr(S) = (0) \quad \text{and} \quad l(M)P = (0) \ .$$

1.3.15. Lemma : If $A \subseteq E$, $B \subseteq E$ such that $A \subseteq B$ then $Ann(B) \subseteq Ann(A)$.

Proof : We have from 1.3.11 that $Ann(B)\,B = (0)$.

Therefore, $Ann(B)\,A \subseteq Ann(B)\,B = (0)$.

$$\Rightarrow \quad Ann(B) \subseteq Ann(A). \quad //$$

1.3.16. <u>Corollary</u> : If $E = N$ then $l(B) \subseteq l(A)$ when $A \subseteq B$.

1.3.17. <u>Lemma</u> : If $P, Q \subseteq N$ such that $P \subseteq Q$ then $r_E(Q) \subseteq r_E(P)$.

<u>Proof</u> : We have from 1.3.11 that $Qr_E(Q) = (0)$.

Therefore, $Pr_E(Q) \subseteq Qr_E(Q) = (0)$

$$\Rightarrow \quad r_E(Q) \subseteq r_E(P) \; . \; //$$

1.3.18. <u>Corollary</u> : If $E = N$ then $r(Q) \subseteq r(P)$ when $P \subseteq Q$.

1.3.19. <u>Lemma</u> : If $S \subseteq N$ and $M \subseteq E$ then

(a) $r_E(\text{Ann}(r_E(S))) = r_E(S)$ and

(b) $\text{Ann}\,(r_E(\text{Ann}(M))) = \text{Ann}(M)$.

<u>Proof</u> : By 1.3.11, we have $S\,r_E(S) = (0)$

$$\Rightarrow \quad S \subseteq \text{Ann}\,(r_E(S)) \qquad \ldots \qquad (i)$$

Thus, replacing S by Ann(M) we get

$$\text{Ann}(M) \subseteq \text{Ann}(r_E(\text{Ann}(M))) \qquad \ldots \qquad (ii)$$

Again by using 1.3.17 in (i) we get

$$r_E(S) \supseteq r_E(\text{Ann}(r_E(S))) \qquad \ldots \qquad (iii)$$

Also, we have from 1.3.11, $\text{Ann}(M)\,M = (0)$.

$$\Rightarrow \quad M \subseteq r_E(\text{Ann}(M)) \qquad \ldots \qquad (iv)$$

Replacing M by $r_E(S)$ we get

$$r_E(S) \subseteq r_E\ (Ann\ (r_E(S))) \quad \ldots \quad (v)$$

Using the result 1.3.15 in (iv), we get

$$Ann(M) \supseteq Ann\ (r_E\ (Ann(M))) \quad \ldots \quad (vi)$$

Therefore, (ii) and (vi) give, $Ann(M) = Ann(r_E(Ann(M)))$.

Also, (iii) and (v) give, $r_E(S) = r_E(Ann(r_E(S)))$. //

1.3.20. Corollary : If E = N then we get

$$r(l(r(S))) = r(S) \text{ and}$$

$$l(r(l(S))) = l(S)\ .$$

1.3.21. Lemma [14] : Let A be a subnear-ring of N and $S \subseteq A$ then $r_N(S) \cap A = r_A(S)$ and $l_N(S) \cap A = l_A(S)$.

Proof : Let $x \in r_A(S)$ then

$$Sx = (0) \quad \text{and} \quad x \in A(\subseteq N)$$

$$\Rightarrow \quad x \in r_N(S) \cap A\ .$$

$$\Rightarrow \quad r_A(S) \subseteq r_N(S) \cap A$$

Conversely, let $y \in r_N(S) \cap A$

$$\Rightarrow \quad y \in r_N(S) \text{ and } y \in A$$

$$\Rightarrow \quad Sy = (0) \quad \text{and} \quad y \in A(\subseteq N)$$

$\Rightarrow \quad y \in r_A(S)$

$\Rightarrow \quad r_N(S) \cap A \subseteq r_A(S)$

Therefore, $r_N(S) \cap A = r_A(S)$

Similarly we can show that $l_N(S) \cap A = l_A(S)$. //

1.3.22. <u>Lemma</u> : If $x \in N$ such that $l(x^t) = l(x^{t+1})$ for some $t \geq 1$ then $l(x^{t+m}) = l(x^t)$ for all positive integer m.

<u>Proof</u> : If $m = 1$ then $l(x^{t+1}) = l(x^t)$ which is our hypothesis. Thus the given result is true for $m = 1$.

Now, let the result is true for $m = r > 1$.

i.e., $l(x^{t+r}) = l(x^t)$. ... (i)

Let $y \in l(x^{t+r+1})$ then $yx^{t+r+1} = 0$.

$\Rightarrow (yx)\ x^{t+r} = 0$

$\Rightarrow \quad yx \in l(x^{t+r}) = l(x^t)$, [by (i)]

$\Rightarrow yx^{t+1} = 0$

$\Rightarrow \quad y \in l(x^{t+1}) = l(x^t)$, by hypothesis.

$\Rightarrow \quad l(x^{t+r+1}) \subseteq l(x^t)$.

But, since r+1 is a positive integer, then clearly $l(x^t) \subseteq l(x^{t+r+1})$. Thus, we have $l(x^{t+r+1}) = l(x^t)$.

Therefore, when the result is true for $m = r$ then it is true for $m = r+1$. Hence, by induction, we have $l(x^{t+m}) = l(x^t)$ for all positive integer m. //

Similarly we get the following

1.3.23. Lemma : If $x \in N$ such that $r(x^t) = r(x^{t+1})$ for some $t \geq 1$ then $r(x^{t+m}) = r(x^t)$ for all positive integer m. //

1.4. Chain conditions, Direct sums and N-homomorphisms.

1.4.1. Definitions : Let $\mathfrak{F}$ be a non-empty collection of subsets of a near-ring N and C be a subclass of $\mathfrak{F}$. Then C is called a chain if, for $D_1, D_2 \in C$, either $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$

Let $D_m \in \mathfrak{F}$ then D_m is called a maximal element of $\mathfrak{F}$ if D_m is not properly contained in any element of $\mathfrak{F}$.

Let $D_m' \in \mathfrak{F}$ then D_m' is called a minimal element of $\mathfrak{F}$ if D_m' contains no element of $\mathfrak{F}$ properly.

1.4.2. Zorn's Lemma : Let $\mathfrak{F}$ be a non-empty collection of subsets of N. If the union of each chain in $\mathfrak{F}$ (with respect to set inclusion as the partial ordering) is an element of $\mathfrak{F}$, then $\mathfrak{F}$ contains a maximal element.

1.4.3. Definitions : (i) A near-ring N is said to satisfy the ascending chain condition (acc) on its left N-subgroups if any ascending sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ of left N-subgroups of N stops after a finite steps (i.e., there exists a $t \in Z^+$ such that $A_t = A_{t+1} = \cdots$).

(ii) A near-ring N is said to satisfy <u>maximum condition</u> on left N-subgroups of it if any family of left N-subgroups of N has a maximal element (w.r.t. set inclusion as its partial ordering).

Analogously, we define the following .

(iii) A near-ring N is said to satisfy the <u>descending chain condition (dcc)</u> on its left N-subgroups if any descending sequence $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ of left N-subgroups of N stops after a finite steps (i.e., there exists a $t \in Z^+$ such that $A_t = A_{t+1} = \ldots$) .

(iv) A near-ring N is said to satisfy <u>minimum condition</u> on left N-subgroups of it if any family of left N-subgroups of N has a minimal element.

The equivalence of (i), (ii) and (iii), (iv) of the above definitions easily follows exactly as in case of well known algebraic structures like group, ring etc.

We note that the acc and the dcc and its equivalent statements on other subalgebraic structures can be defined analogously.

1.4.4. <u>Lemma</u> : In a near-ring N, the maximum condition on left(right) annihilators is equivalent to the minimum condition on right (left) annihilators.

<u>Proof</u> : Let the maximum condition on left annihilators in N hold . Now, we consider an infinite descending chain on right annihilators as

$$r(S_1) \supseteq r(S_2) \supseteq r(S_3) \supseteq \cdots \qquad \text{(i)}$$

where S_1, S_2, S_3, ..., are subsets of N.

Thus, by 1.3.16, we get

$$l(r(S_1)) \subseteq l(r(S_2)) \subseteq l(r(S_3)) \subseteq \cdots \qquad \text{(ii)}$$

Since the maximum condition holds on left annihilators in N and so the chain (ii) will terminate after a finite steps. Thus, for finite $t \in Z^+$, we have

$$l(r(S_t)) = l(r(S_{t+1})) = l(r(S_{t+2})) = \cdots$$

$$\Rightarrow \quad r(l(r(S_t))) = r(l(r(S_{t+1}))) = r(l(r(S_{t+2}))) = \cdots$$

$$\Rightarrow \quad r(S_t) = r(S_{t+1}) = r(S_{t+2}) = \cdots, \qquad \text{[by 1.3.20]}$$

Hence the descending chain (i) also terminates after a finite steps. So the minimum condition on right annihilators holds when the maximum condition on left annihilators holds. //

1.4.5. Lemma : Let N be a near-ring satisfying the acc(dcc) on right annihilators in N and S be a subnear-ring of N then S also satisfies the acc (dcc) on right annihilators in S.

Proof : Let the ascending chain on right annihilators of A_1, A_2, A_3, ... be

$$r_S(A_1) \subseteq r_S(A_2) \subseteq r_S(A_3) \subseteq \cdots \qquad \text{(i)}$$

where A_1, A_2, A_3, ... are subsets of N.

Hence 1.3.16 gives,

$$l_S(r_S(A_1)) \supseteq l_S(r_S(A_2)) \supseteq l_S(r_S(A_3)) \supseteq \cdots$$

Again, by 1.3.17, we get

$$r_N(l_S(r_S(A_1))) \subseteq r_N(l_S(r_S(A_2))) \subseteq r_N(l_S(r_S(A_3))) \subseteq \cdots$$

But N satisfies the acc on right annihilators in N. Thus, for a finite $t \in Z^+$, we have

$$r_N(l_S(r_S(A_t))) = r_N(l_S(r_S(A_{t+1}))) = \cdots$$

$$\Rightarrow \quad r_N(l_S(r_S(A_t))) \cap S = r_N(l_S(r_S(A_{t+1}))) \cap S = \cdots$$

$$\Rightarrow \quad r_S(l_S(r_S(A_t))) = r_S(l_S(r_S(A_{t+1}))) = \cdots$$

$$\Rightarrow \quad r_S(A_t) = r_S(A_{t+1}) = \cdots \quad , \quad \text{[by 1.3.20]}$$

Hence the subnear-ring S of N satisfies the acc on right annihilators in S. //

Similarly, if the near-ring N satisfies the acc (dcc) on left annihilators in N then the subnear-ring S of N also satisfies the acc(dcc) on left annihilators in S.

1.4.6. Definitions : Let $\{E_i\}$ be any family of N-groups then $\{(\ldots, e_i, \ldots) \mid e_i \in E_i, \ e_i = 0$ for all but a finite number of $i\}$ is defined as the direct sum of E_i's and is denoted by $\bigoplus E_i$. Each E_i is called a direct summand of $\bigoplus E_i$.

Also, if $\{N_i\}$ is a family of near-rings then

$\{(\ldots, n_i, \ldots) \mid n_i \in N_i, n_i = 0$ for all but a finite number of $i\}$ is defined as the direct sum of N_i's and is denoted by $\bigoplus N_i$. Each N_i is called a direct summand of $\bigoplus N_i$.

1.4.7. Lemmas : (i) Direct sum $\bigoplus_{i=1}^{t} E_i$ of N-groups is an N-group if the elements of $\bigoplus_{i=1}^{t} E_i$ are added coordinatewise and for $n \in N$, $e_i \in E_i$,

$$n(\ldots, e_i, \ldots) = (\ \ldots, ne_i, \ldots),\ i = 1,2,\ldots, t.$$

(ii) Direct sum $\bigoplus_{i=1}^{t} N_i$ of near-rings is a near-ring if the elements of $\bigoplus_{i=1}^{t} N_i$ are added coordinatewise and the multiplication defined by $(a_1, a_2, \ldots, a_t)(b_1, b_2, \ldots, b_t)=(a_1b_1, a_2b_2, \ldots, a_tb_t)$ for $a_i, b_i \in N_i$, $i = 1,2,\ldots, t$.

1.4.8. Definition : A sum $A = \sum_{i=1}^{t} A_i$ for a finite number of ideals (subgroups) A_i of a near-ring N is called a direct sum of ideals (subgroups) provided each element of A is uniquely expressible in the form $\sum_{i=1}^{t} a_i$, where $a_i \in A_i$, $i = 1,2,\ldots, t$ and this direct sum is denoted by $A = A_1 \oplus A_2 \oplus \ldots \oplus A_t$.

1.4.9. Lemma : Let $A_i (i = 1,2,\ldots, t)$ be ideals (subgroups) of a near-ring N then $\sum_{i=1}^{t} A_i$ is a direct sum of ideals (subgroups) if and only if $A_j \cap \sum_{i \neq j}^{t} A_i = (0)$.

1.4.10. Definitions : Let N and N_1 be two near-rings then a mapping $f : N \to N_1$ is called a homomorphism if for all $x,y \in N$,

$$f(x+y) = f(x) + f(y) \quad \text{and}$$

$$f(xy) = f(x)\, f(y).$$

Kernel of homomorphism f(ker f) = $\{x \in N \mid f(x) = 0\}$.

If Ker f = (0) then f is called a monomorphism and N is said to be embedded in N_1. If homomorphism f is injective as well as surjective then f is called an isomorphism and we write $N \cong N_1$.

1.4.11. Definitions : If M and M_1 are two N-groups over the near-ring N then a mapping $f : M \to M_1$ is called an N-homomorphism if for all $x,y \in M$, $n \in N$, $f(x+y) = f(x) + f(y)$ and $f(nx) = nf(x)$.

Kernel of N-homomorphism f(Ker f) = $\{x \in M \mid f(x) = 0\}$. If Ker f = (0) then f is called an N-monomorphism and M is said to be embedded in M_1. When M is embedded in M_1 then we can consider M as an N-subgroup of M_1. An N-homomorphism f is called an N-epimorphism if f is surjective. If N-homomorphism f is injective as well as surjective then f is called an N-isomorphism. It is denoted by $M \cong M_1$.

CHAPTER II

fsd-1 near-ring groups with chain conditions

We recall that a module M is said to have finite Goldie dimension if it does not contain an infinite direct sum of submodules. This is equivalent to saying that for any increasing sequence of submodules of M, such as $V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots$, there is an $i \in Z^+$ such that V_i is essential in V_j, for all $j \geq i$.

Among the several possible ways of dualising this concept, we extend the concept of P.Fleury [18] what is known as finite spanning dimension. We note that the basis of a vector space can be defined as a maximal set of linearly independent vectors or as a minimal set of vectors which spans the space. The generalisation of the first one coincides with the concept of what is known as Goldie dimension. We confine our discussion to the second description of a basis of a vector space. It is to be noted that the generalisation of second one in case of a near-ring group coincides with what we have called an N-group with finite spanning dimension. The main results of this chapter have appeared in our paper [[15]], IJPAM.

This chapter consists of five sections where the first one (prerequisites) contains results on small ideals of N-groups.

The second section contains properties of essential ideals and essential N-subgroups of an N-group E. We prove here the generalised form of a result of Oswald [41] viz; Let N-group E be with acc on annihilators such that $Z_1(E) = (0)$. If N has no infinite direct sum of left ideals and every essential left ideal of N is an essential N-subgroup of ${}_N N$ then N satisfies the dcc on annihilators of subsets of E.

The third section contains the results on the family $\mathcal{A}(E)$ of associated strictly primes of E such as $\mathcal{A}(E_1 \oplus E_2) = \mathcal{A}(E_1) \cup \mathcal{A}(E_2)$ where E_1, E_2 are N-groups. Also, for two N-subgroups M, Q and an ideal T of E we have $\mathcal{A}((M \cap Q)/_T) = \mathcal{A}(M/_T \cap Q/_T) \subseteq \mathcal{A}(M/_T) \cap \mathcal{A}(Q/_T)$ where $T \subseteq M, Q$.

In the fourth section, introducing the notion of finite spanning dimension (fsd-1) in an N-group which is different from that in Satyanarayana and Coworkers [44, 47], we prove our main results of the chapter as follows :

Every fsd-1 N-group E determines a unique integer $t(>0)$ (termed as finite spanning dimension Sd_1) such that E is a sum $\sum_{i=1}^{t} E_i$ of hollow ideals with $\sum_{j \neq i}^{t} E_j \neq E$ $(1 \leq i, j \leq t)$.

Every fsd-1 N-group E satisfies the acc on its supplements (in some special cases).

If E is an fsd-1 N-group and H is a non-zero supplement in E then $E/_H$ satisfies the dcc on ideals of $E/_H$.

The last section - the fifth one, contains results on a prime complete N-group E as follows :

If E is an fsd-1 N-group with non-small extension satisfying the acc on annihilators then

(i) $\mathcal{A}(E)$ is a finite collection.

(ii) there exists an s-p decomposition of (0) in E and

(iii) if $E_1 \cap E_2 \cap \ldots \cap E_t = (0)$ is an s-p decomposition of (0) in E then $\mathcal{A}(E) = \bigcup_{i=1}^{t} \mathcal{A}(E/_{E_i})$

Another interesting result established here shows how an fsd-1 N-group depends on the fgd character of the attached near-ring so as to be more well behaved as in case of a Noetherian module. That is, if N is a commutative fgd ring with 1 and $Z_1(E) = (0)$ then (in some special cases) for any $x \in \bigcap_{P \in \mathcal{A}(E)} P$, we have a $t \in Z^+$ such that $x^t \in \text{Ann}(E)$.

2.1. Prerequisites :

2.1.1. Definitions : Let A and B be two ideals of E such that $A \subset B$ then A is said to be <u>small in B</u> (denoted as $A \subset_s B$) if each ideal L ($\subseteq B$) of E, A + L = B implies L = B. An ideal M of E is small in E if M is small in E when E is considered as an ideal of itself. Obviously, (0) is small in E ($\neq 0$).

2.1.2. <u>Definition</u> : If M is an ideal of E then an N-epimorphism $f : M \to E$ is called <u>a small N-epimorphism</u> if Ker f is small in M.

We write : If M is an ideal of E, then $Hom_1(M,E) = \{f : M \to E \mid$ N-homomorphic image $f(M)$ is an ideal of E$\}$. It is noted that $Hom_1(M,E) \neq \emptyset$.

2.1.3. <u>Lemma</u> : If T is a proper ideal of E then the following statements are equivalent.

(a) $T \subset_s E$

(b) The natural map $f : E \to E/_T$ is a small N-epimorphism.

(c) For every ideal M of E and for every $h \in Hom_1(M,E)$, $Im\ h + T = E \Rightarrow Im\ h = E$.

<u>Proof</u> : (a) $\Rightarrow$ (b) :

Let the natural N-epimorphism

$f : E \to E/_T$ defined by $f(e) = e+T$, for $e \in E$,

So, Ker f = T. And $T \subset_s E \Rightarrow f$ is a small N-epimorphism in E.

(b) $\Rightarrow$ (c) :

Let f be a small N-epimorphism in E. So, Ker $f \subset_s E$. But Ker f = T(as above), hence $T \subset_s E$.

Now, let $\text{Im } h + T = E$ where $\text{Im } h$ is an ideal of E as $h \in Hom_1(M,E)$. Since $T \subset_s E$, $\text{Im } h = E$.

(c) $\Rightarrow$ (a) :

Let L be an ideal of E such that $T + L = E$.

Consider the natural inclusion map

$\mathring{i} : L \to E$ such that $\mathring{i}(x) = x$, for $x \in L$

then $\mathring{i}(L) = L \subseteq E$ and so $\mathring{i} \in Hom_1(L,E)$.

Therefore, $T + L = E$ gives $T + \mathring{i}(L) = E$. Hence by (c), $\mathring{i}(L) = E$ which gives $L = \mathring{i}(L) = E$.

Hence $T \subset_s E$. //

2.1.4. <u>Lemma</u> : If T, M and H are ideals of E such that $T \subseteq M \subset E$ then

(a) $M \subset_s E$ if and only if $T \subset_s E$ and $M/_T \subset_s E/_T$.

(b) $(H + T) \subset_s E$ if and only if $H \subset_s E$ and $T \subset_s E$.

<u>Proof</u> : Let $M \subset_s E$ and L be an ideal of E such that $T+L = E$.

Then $M + L \supseteq T + L = E$

$\Rightarrow$ $M + L = E$.

$\Rightarrow$ $L = E$ (for $M \subset_s E$)

$\Rightarrow$ $T \subset_s E$

Again, as $M \subset E$, $M/_T \subset E/_T$ is such that

$$M/_T + L/_T = E/_T$$

$\Rightarrow \quad (M+L)/_T = E/_T$

$\Rightarrow \quad M + L = E$

$\Rightarrow \quad L = E \quad (\text{for } M \subset_s E)$

$\Rightarrow \quad L/_T = E/_T$

Therefore, $M/_T \subset_s E/_T$.

Conversely, assume that $T \subset_s E$ and $M/_T \subset_s E/_T$.

Now, if L is an ideal of E such that

$M + L = E$, then

$M + L + T = E + T = E$

$\Rightarrow \quad M/_T + (L + T)/_T = E/_T \quad (\text{as } T \subseteq M \subset E)$

$\Rightarrow \quad (L + T)/_T = E/_T \quad (\text{for } M/_T \subset_s E/_T)$

$\Rightarrow \quad L + T = E$

$\Rightarrow \quad L = E \quad (\text{for } T \subset_s E)$.

Thus, $M + L = E \Rightarrow L = E$.

Hence $M \subset_s E$. //

(b) Assume $H + T \subset_s E$ ($H + T$ is an ideal of E for H,T are ideals of E).

If L is an ideal of E such that

$H + L = E$, then

$H + L + T = E + T = E$

$\Rightarrow$ $(H+T) + L = E$ (for H,T,L are ideals of E)

$\Rightarrow$ $L = E$ (as $H + T \subset_s E$.)

$\Rightarrow$ $H \subset_s E$.

Similarly we can show that $T \subset_s E$

Conversely, assume $H \subset_s E$ and $T \subset_s E$.

Let L be an ideal of E such that

$H + T + L = E$, then

$T + L = E$ (for $H \subset_s E$)

$\Rightarrow$ $L = E$ (for $T \subset_s E$)

Therefore, $H + T \subset_s E$. //

2.1.5. <u>Lemma</u> : If T, M and L are ideals of E such that $T \subset_s M \subseteq L$ then $T \subset_s L$.

<u>Proof</u> : Let X ($\subseteq L$) be an ideal of E such that

$T + X = L$

$\Rightarrow (T + X) \cap M = L \cap M$

$\Rightarrow T + (X \cap M) = M$, (for $T \subset M \subseteq L$ and 1.2.36)

$\Rightarrow X \cap M = M$, (as $T \subset_s M$)

$\Rightarrow T \subset M = X \cap M \subseteq X$.

Therefore, $L = T + X = X$, (for $T \subset X$)

Thus, $T \subset_s L$. //

2.1.6. <u>Corollary</u> : If T and M are ideals of E such that $T \subset_s M$ then $T \subset_s E$.

Proof : Considering L = E in 2.1.5, we get the required result. //

2.2. Essential ideals and essential N-subgroups.

2.2.1. Definitions : Let A and B be two N-subgroups of E such that $A \subseteq B$ then A is said to be an essential N-subgroup of B (denoted $A \subseteq_e B$) if any N-subgroup C ($\neq 0$) of B has non-zero intersection with A. When $A \subseteq_e B$, we say B is an essential extension of A in E. Here an essential left N-subgroup A of N will mean an essential N-subgroup of $_N N$.

An ideal M of E is called an essential ideal of E (denoted $M \subseteq_e E$) if for any ideal C ($\neq 0$) of E, $M \cap C \neq (0)$. If a left ideal A of N is an essential ideal of $_N N$ then A is called an essential left ideal of N.

2.2.2. Example (J(22), Page 342-343 [42]) :

The group N = {0,1,2,3,4,5,6,7} under addition modulo 8 is an N-group w.r.t. the multiplication defined by the following table

.	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	2	0	4	4	2
2	0	0	0	4	0	0	0	4
3	0	0	0	6	0	4	4	6
4	0	0	0	0	0	0	0	0
5	0	0	0	2	0	4	4	2
6	0	0	0	4	0	0	0	4
7	0	0	0	6	0	4	4	6

Table : 2.1.

N-group ${}_N N$ has two non trivial N-subgroups $\{0,4\}$ and $\{0,2,4,6\}$. Hence each of them has non zero intersection with other N-subgroups of ${}_N N$ and so each of them is an essential N-subgroup of ${}_N N$. Also $\{0,4\} \subseteq_e \{0,2,4,6\}$ which shows the validity of the following lemma.

2.2.3. Lemma : If A,B,C are N-subgroups of E such that $A \subseteq B \subseteq C$ then $A \subseteq_e B \subseteq_e C$ if and only if $A \subseteq_e C$.

Proof : Let P be a non-zero N-subgroup of E such that $P \subseteq C$. Since $B \subseteq_e C$, $B \cap P \neq (0)$.

Also, $B \cap P \subseteq B$ and $A \subseteq_e B$. So $(B \cap P) \cap A \neq (0)$.

Therefore, $P \cap A \supseteq (B \cap P) \cap A \neq (0)$.

Hence $A \subseteq_e C$

Conversely, let $A \subseteq_e C$. Then $A \cap B \neq (0)$, (for $B \subseteq C$).

If M is a non-zero N-subgroup of E such that $M \subseteq B \subseteq C$ then, by 1.2.25, M is a non-zero N-subgroup of C. Since $A \subseteq_e C$, it follows that $A \cap M \neq (0)$ which gives $A \subseteq_e B$.

Again, if H is any non-zero N-subgroup of E with $H \subseteq C \subseteq E$ then $A \cap H \neq (0)$, (for $A \subseteq_e C$).

So, $A \subseteq B \Rightarrow (0) \neq A \cap H \subseteq B \cap H$.

Thus, $B \subseteq_e C$. //

2.2.4. Lemma [[15]] : Let A and B be two N-subgroups of E such that $B \subseteq_e A$. If $a\ (\neq 0) \in A$ then there exists an essential

N-subgroup L of ${}_N N$ such that $La \subseteq B$ $La \neq (0)$.

Proof : Write $L = \{n \in N \mid na \in B\}$. Clearly, $La \subseteq B \subseteq A$ and $Na \subseteq A$ as A is an N-subgroup of N, $a \in A$.

Since $1 \in N$, $Na \neq (0)$. Again, $B \subseteq_e A$ gives $B \cap Na \neq (0)$.

Let $(0 \neq)$ $b \in B \cap Na$. Then $b = na$ (say) for $n \in N$. Thus $b = na \in B$ which gives $n \in L$. Hence $b = na \in La$.

Therefore, $La \neq (0)$ (for $b \neq 0$).

Now, let $x, y \in L$ then $xa, ya \in B$.

So, $(x - y)\, a = xa - ya \in B$.

$\Rightarrow$ $x - y \in L$. ... (i)

Also, since B is an N-subgroup of E, for $n \in N$, $(nx)\, a = n(xa) \in B$ (for $xa \in B$).

Therefore, $nx \in L$ ... (ii)

Thus L is an N-subgroup of ${}_N N$.

Again, for an N-subgroup I $(\neq 0)$ of ${}_N N$,

$Ia = (0)$

$\Rightarrow$ $Ia \subseteq B$

$\Rightarrow$ $I \subseteq L$

$\Rightarrow$ $L \cap I = I \neq (0)$

And, $Ia \neq (0)$

$\Rightarrow$ $B \cap Ia \neq (0)$, (for Ia is an N-subgroup of A and $B \subseteq_e A$).

Now, let $(0 \neq)$ $x \in B \cap Ia$ then $x = b = \alpha a$ for $b \in B$, $\alpha \in I$.

Then $\alpha a \in B$

$\Rightarrow \alpha \in L$, (by choice of L)

$\Rightarrow \alpha \in L \cap I$.

Now, $\alpha = 0 \Rightarrow x = 0$, a contradiction.

So, $L \cap I \neq (0)$.

Therefore, L is an essential N-subgroup of ${}_N N$ such that $La \subseteq B$ and $La \neq (0)$. //

In an N-group E, the subset $Z_1(E) = \{u \in E \mid Lu = (0)$, for some essential N-subgroup L of ${}_N N\}$ plays an important role in what follows.

2.2.5. __Lemma__ : For an $x \in E$, Ann(x) is an essential N-subgroup of ${}_N N$ if and only if $x \in Z_1(E)$.

__Proof__ : Let Ann(x) be an essential N-subgroup of ${}_N N$. Then, by 2.2.4, for $1 \in {}_N N$ there exists an essential N-subgroup L of ${}_N N$ such that $L.1 \neq (0)$ and $L.1. \subseteq \text{Ann}(x)$.

So, $L \subseteq \text{Ann}(x)$

$\Rightarrow Lx = (0)$

$\Rightarrow x \in Z_1(E)$.

Conversely, let $x \in Z_1(E)$ then $Ax = (0)$ for some essential N-subgroup A of ${}_N N$. Thus $A \subseteq \text{Ann}(x) \subseteq {}_N N$. Hence by 2.2.3, Ann(x) is an essential N-subgroup of ${}_N N$. //

2.2.6. __Lemma__ [[15]] : If I is an N-subgroup of ${}_N N$ and for $B \subseteq E$, $\text{Ann}(B) \subseteq_e I$ and $Z_1(E) = (0)$ then $\text{Ann}(B) = I$.

Proof : Let ($0 \neq$) $x \in I$ then by 2.2.4, there exists an essential N-subgroup L of ${}_N N$ such that $Lx \neq (0)$, $Lx \subseteq Ann(B)$.

So, $(Lx)\, r_E(Ann(B)) \subseteq Ann(B)\, r_E(Ann(B)) = (0)$ [by 1.3.11]

$\Rightarrow$ $L(x\, r_E\, (Ann(B)) = (0)$

$\Rightarrow$ $x\, r_E(Ann(B)) \in Z_1(E) = (0)$

$\Rightarrow$ $x \in Ann\, (r_E\, (Ann(B))) = Ann(B)$, [by 1.3.19(b)]

$\Rightarrow$ $I \subseteq Ann\, (B)$

Now considering the hypothesis, we get $Ann(B) = I$. //

2.2.7. Definitions : An N-group E is said to be an N-group with acc on annihilators if any ascending chain $Ann(M_1) \subseteq Ann(M_2) \subseteq Ann(M_3) \subseteq \ldots$ of annihilators of subsets $M_1, M_2, M_3, \ldots$ of E stops after a finite steps. Similarly, we can define an N-group E with dcc on annihilators for the descending chain.

2.2.8. Lemma [[15]] : Let E be with acc on annihilators such that $Z_1(E) = (0)$. If N has no infinite direct sum of left ideals and every essential left ideal of N is an essential N-subgroup of ${}_N N$ then N satisfies the dcc on annihilators of subsets of E.

Proof : Let X and Y be subsets of E such that $B = Ann(X)$ and $C = Ann(Y)$. Thus, B,C are N-subgroups of ${}_N N$ [by 1.3.3].

Now, if $B \subset C$ and B is an essential N-subgroup of C then by 2.2.6, $B = C$ as $B = Ann(X)$. Hence B is not an essential

N-subgroup of C. So, there exists an N-subgroup D($\neq 0$) of ${}_N N$ such that $D \subseteq C$, $B \cap D = (0)$.

Let $A_1 \supset A_2 \supset A_3 \supset \ldots$ be a strictly descending chain of annihilators of subsets of E. Since $A_i \supset A_{i+1}$, by the above argument, there exists an N-subgroup P_i ($\neq 0$) of ${}_N N$ such that

$P_i \subseteq A_i$ and $A_{i+1} \cap P_i = (0)$ ... (i)

Consider $M = \{X_m\}$, the family of all left ideals of N such that $A_{i+1} \cap X_m = (0)$. The union of each chain of M is again a left ideal in M and satisfies the condition $A_{i+1} \cap X_m = (0)$. Thus, by Zorn's Lemma in 1.4.2., M has a maximal element X_i(say) such that $A_{i+1} \cap X_i = (0)$ ... (ii)

Again, A_{i+1} and X_i being left ideals of N, $A_{i+1} + X_i$ is also a left ideal of N.

Now, let V be a left ideal of N such that $(A_{i+1} + X_i) \cap V = (0)$.

Now, $a_{i+1} = x_i + v$, for some $a_{i+1} \in A_{i+1}$, $x_i \in X_i$, $v \in V$.

$\Rightarrow$ $v = -x_i + a_{i+1} \in X_i + A_{i+1} = A_{i+1} + X_i$

$\Rightarrow$ $v \in (A_{i+1} + X_i) \cap V = (0)$

$\Rightarrow$ $a_{i+1} = x_i \in A_{i+1} \cap X_i = (0)$

$\Rightarrow$ $A_{i+1} \cap (X_i + V) = (0)$

Since X_i is maximal with condition $A_{i+1} \cap X_i = (0)$, it follows that $X_i + V = X_i$ as $X_i \subseteq X_i + V$. This gives $V \subseteq X_i$ and so $V = V \cap X_i \subseteq V \cap (A_{i+1} + X_i) = (0)$.

Thus, $A_{i+1} + X_i$ is an essential left ideal of N such that $A_{i+1} \cap X_i = (0)$ and the assumed hypothesis gives that $A_{i+1} + X_i$ is an essential N-subgroup of ${}_N N$. And so for P_i, chosen above, $P_i \cap (A_{i+1} + X_i) \neq (0)$.

Suppose, $\alpha = p_i = a_{i+1} + x_i$, for $p_i \in P_i$, $a_{i+1} \in A_{i+1}$, $x_i \in X_i$. Then, $x_i = -a_{i+1} + p_i \in A_{i+1} + P_i \subseteq A_i + P_i$, for $A_{i+1} \subset A_i$. So, $x_i \in A_i$ (for $P_i \subseteq A_i$) which gives $x_i \in A_i \cap X_i$.

Now, if $x_i = 0$ then $p_i \in A_{i+1}$ which gives $p_i \in A_{i+1} \cap P_i = (0)$. So, $p_i = 0$.

Therefore, $P_i \cap (A_{i+1} + X_i) = (0)$ and this is a contradiction. Hence $x_i \neq 0$ and therefore $A_i \cap X_i \neq (0)$.

Let $C_i = A_i \cap X_i$, a non-zero left ideal of N.

Then, $C_i \cap A_{i+1} = (A_i \cap X_i) \cap A_{i+1}$

$= (A_{i+1} \cap A_i) \cap X_i$

$= A_{i+1} \cap X_i$, (as $A_i \supset A_{i+1}$)

$= (0)$, [by (ii)]

Therefore, when $A_i \supset A_{i+1}$, we get a non-zero ideal $C_i = A_i \cap X_i$ such that $C_i \cap A_{i+1} = (0)$ . . . (iii)

Now, for different values of i, we get an infinite family $\{C_1, C_2, C_3, \ldots\}$ of non-zero left ideals of N such that (iii) holds.

Also, $C_i = A_i \cap X_i \subseteq A_i$ ... (iv)

Therefore, $C_1 \cap C_2 \subseteq C_1 \cap A_2 = (0)$, [by (iii) and (iv)]

Again, $C_1 \cap (C_2 + C_3) \subseteq C_1 \cap (A_2 + A_3)$, [by (iv)]

$\subseteq C_1 \cap A_2$, as $A_2 \supset A_3$

$= (0)$, [by (iii)]

$\Rightarrow C_1 \cap (C_2 + C_3) = (0)$ ... (v)

And if $x \in C_2 \cap (C_1 + C_3)$ then

$x = c_2 = c_1 + c_3$, for $c_i \in C_i$, $i = 1,2,3$.

$\Rightarrow c_1 = c_2 - c_3 \in C_2 + C_3$

So, $c_1 \in C_1 \cap (C_2 + C_3) = (0)$, [by (v)]

$\Rightarrow$ $c_1 = 0$ and $c_2 = c_3 \in C_3$.

$\Rightarrow$ $c_2 \in C_2 \cap C_3 \subseteq C_2 \cap A_3 = (0)$, [by (iii) and (iv)]

$\Rightarrow$ $c_2 = 0$ and hence $C_2 \cap (C_1 + C_3) = (0)$.

Similarly, $C_3 \cap (C_1 + C_2) = (0)$. Thus $C_1 \oplus C_2 \oplus C_3$ is a direct sum of non-zero left ideals of N.

Proceeding in this way, we find an infinite direct sum $C_1 \oplus C_2 \oplus C_3 \oplus \ldots$ of nonzero left ideals of N. This goes against our hypothesis and hence there exists a $t \in Z^{+}$ such that $A_t = A_{t+1} = A_{t+2} = \dots$ Therefore, N satisfies dcc on annihilators of subsets of E. //

2.3. The Associated Primes of N-groups.

2.3.1. Definitions [[15]] : A non-zero N-subgroup M of E is said to be a prime N-subgroup of E if for each non-zero N-subgroup T of M, Ann(T) = Ann(M). If E is a prime N-subgroup of itself then E is called a prime N-group. It is noted that (0) is not a prime N-subgroup of E.

Example of such a prime N-group is given below.

2.3.2. Example (H(20), page 341-342 [42]) :

N = {0,a,b,c,x,y} is a near-ring under the operations addition [defined in table : 1.3(i)] and multiplication defined by the following table.

.	0	a	b	c	x	y
0	0	0	0	0	0	0
a	0	a	a	a	0	a
b	0	b	b	b	0	b
c	0	c	c	c	0	c
x	0	x	x	x	0	x
y	0	y	y	y	0	y

Table : 2.2.

This N-group ${}_N N$ has N-subgroups $\{0,a\}$, $\{0,b\}$, $\{0,c\}$, $\{0,x,y\}$ and ${}_N N$. Annihilator of each of the above subgroups is $\{0\}$. So N-group ${}_N N$ is a prime N-group. //

2.3.3. Lemma : If N is d.g. then Ann(M) of a prime N-subgroup M of E is such that for any invariant subnear-rings B and C of N, $BC \subseteq Ann(M)$ implies either $B \subseteq Ann(M)$ or $C \subseteq Ann(M)$.

Proof : We have from 1.3.5. that Ann(M) is an ideal of N.

Now, if $C \not\subseteq Ann(M)$ then $CM \neq (0)$.

But $(BC)M = (0)$ as $BC \subseteq Ann(M)$

$\Rightarrow B(CM) = (0)$

$\Rightarrow B \subseteq Ann(CM)$. ... (i)

Again, $CM \subseteq M$ and CM is an N-subgroup of M. For this, let $\sum_{fin} c_i m_i \in CM$, for $c_i \in C$, $m_i \in M$. Obviously, CM is a subgroup of M. Now, for $n \in N$, we have $n \sum_{fin} c_i m_i = (\sum_j s_j)m$, where $m = \sum_{fin} c_i m_i \in M$ and $n = \sum_j s_j$, s_j's are distributive elements of N.

Therefore, $n \sum_{fin} c_i m_i = (\sum_j s_j)m$

$= \sum_j (s_j m)$, as M is an N-subgroup of E.

$= \sum_j [\sum_i s_j(c_i m_i)]$, as s_j is distributive

$$= \sum_j [\sum_i (s_j c_i) m_i], \text{ where } s_j c_i \in C.$$

$$= \sum_{fin} c' m', \text{ where } c' \in C, \ m' \in M.$$

Thus, $n \sum_{fin} c_i m_i \in C\,M$. So CM is a non-zero N-subgroup of M. But M being a prime N-subgroup of E, it follows that Ann(M) = Ann(CM).

So, $B \subseteq \text{Ann}(CM) \Rightarrow B \subseteq \text{Ann}(M)$. //

Now we give the following.

2.3.4. Definition : An ideal I of N is called a strictly prime ideal of N if for non-zero invariant sub near-rings A and B of N, $AB \subseteq I$ implies either $A \subseteq I$ or $B \subseteq I$.

Thus we immediately have the following result from 2.3.3.

2.3.5. Lemma : If N is a d.g.nr. and M is a prime N-subgroup of E then Ann(M) is a strictly prime ideal of N. /

Let M be a prime N-subgroup of E. If M_1 is any N-subgroup of M then any N-subgroup M_2 of M_1 is also an N-subgroup of M. Therefore, M being prime, $\text{Ann}(M_2) = \text{Ann}(M_1) = \text{Ann}(M)$. So, M_1 is a prime N-subgroup of M. Thus, we get

2.3.6. Lemma : If M is a prime N-subgroup of E then any N-subgroup M_1 of M is again a prime N-subgroup of M.

Moreover, if M_1 is a prime N-subgroup of M then clearly M_1 is a prime N-subgroup of E. //

2.3.7. Theorem [[15]] : An ideal P of a d.g.nr. N is a strictly prime ideal if and only if the near-ring $N/_P$ is a prime N-group.

Proof : Let $N/_P$ be a prime N-group. Also, let B and C be two invariant subnear-rings of N such that $CB \subseteq P$.

Now, let $B \not\subseteq P$ then $C(B+P) \subseteq CB + P$.

For, if $c \in C$, $b \in B$, $p \in P$ then $c(b+p) \in C(B+P)$.

Thus, $c(b+p) = (\Sigma s_i)(b+p)$, where $c = \Sigma s_i$, for distributive elements s_i of N.

$$
\begin{aligned}
\text{Hence}\quad c(b+p) &= \Sigma(s_i(b+p)) \\
&= \Sigma (s_i b + s_i p) \\
&= \Sigma (s_i b + p_i), \text{ where } p_i = s_i p \in P. \\
&= \Sigma (s_i b) + p', \text{ for some } p' \in P \text{ as P is an ideal of N.} \\
&= (\Sigma s_i)\, b + p' \\
&= cb + p' \in CB + P \\
&\Rightarrow C(B+P) \subseteq CB + P \subseteq P+P, \quad \text{(by supposition).} \\
&\Rightarrow C(B + P) \subseteq P \\
&\Rightarrow C(B + P)/_P = P
\end{aligned}
$$

$$\Rightarrow \quad (C(B + P))/_P = P$$

$$\Rightarrow \quad C \subseteq Ann\,((B + P)/_P\,) \qquad \ldots \qquad (i)$$

Now, B being a left N-subgroup of N and P being a left ideal of N, by 1.2.28, B+P is a left N-subgroup of N. Thus B+P is an N-subgroup of ${}_N N$. Hence by 1.2.39, $(B+P)/_P$ is an N-subgroup of the N-group $N/_P$.

But $N/_P$ is a prime N-group and hence $Ann(N/_P) = Ann\,((B + P)/_P)$. Thus (i) gives, $C \subseteq Ann\,(N/_P)$ ... (ii)

Now, $x \in Ann\,(N/_P) \Rightarrow x\,(N/_P) = P$

$$\Rightarrow \quad xN \subseteq P$$

$$\Rightarrow \quad x \in P, \quad \text{for } 1 \in N.$$

Thus, $Ann\,(N/_P) \subseteq P$. So (ii) gives, $C \subseteq P$.

Hence P is a strictly prime ideal of N.

Conversely, let P be strictly prime ideal of N. Now, if $M/_P (\neq P)$ is an N-subgroup of $N/_P$ then by 1.2.39, M is an N-subgroup of ${}_N N$, such that $P \subset M \subseteq N$ and by 1.3.5. $Ann\,(M/_P)$ is an ideal of $N/_P$. Hence $Ann\,(M/_P)$ is an invariant subnear-ring of N.

Again, $x \in Ann(M/_P) \Rightarrow x\,(M/_P) = P$

$$\Rightarrow \quad xM \subseteq P$$

$$\Rightarrow \quad (Ann(M/_P))\, M \subseteq P.$$

Therefore, $\text{Ann}(M/_P)\ (MN) = (\text{Ann}(M/_P)M)\ N \subseteq PN \subseteq P$ as P is an ideal of N.

So, $\text{Ann}(M/_P)(MN) \subseteq P$ ... (iii)

Also, since M is a left N-subgroup of N, MN is a left as well as a right N-subgroup of N. Hence MN is an invariant subnearring of N.

Now, as $1 \in N$, $M \subseteq MN$ and therefore it follows that $M \subseteq P$ if $MN \subseteq P$. And this is not true for $M/_P \neq P$.

So, $MN \nsubseteq P$ and P being a strictly prime ideal of N, it follows from (iii) that $\text{Ann}(M/_P) \subseteq P$.

As P is an ideal of N, we have $PM \subseteq P$ and $P(M/_P) = P$.

So $P \subseteq \text{Ann}\ (M/_P)$. Hence $\text{Ann}\ (M/_P) = P$.

Similarly, we get $\text{Ann}(N/_P) = P$.

Thus, $\text{Ann}\ (N/_P) = P = \text{Ann}\ (M/_P)$

Therefore, $N/_P$ is a prime N-group. //

In what follows N will mean a d.g.nr. with 1.

2.3.8. <u>Definitions</u> [[15]] : The collection $\mathcal{A}(E) = \{P | P = \text{Ann}(M)$, for some prime N-subgroup M of E$\}$ is said to be the <u>family of associated strictly primes</u> of E. Clearly, $\mathcal{A}(0) = \emptyset$.

An N-group E is called <u>primary</u> if $\mathcal{A}(E)$ is singleton. Thus, a prime N-group is always primary.

The following example shows that the converse is not always true.

2.3.9. Example [[15]] : Consider Z_9(= E) the Z-group of integers addition modulo 9.

Here Ann(E) = ⟨9⟩ and G = $\{\bar{0},\bar{3},\bar{6}\}$ is a Z-subgroup of E where Ann(G) = ⟨3⟩. Clearly, Ann(E) ≠ Ann(G). So E is not a prime Z-group.

On the otherhand, G being of prime order, it has no non-zero proper Z-subgroup. So, G is a prime Z-subgroup of E. Moreover, G is the only sub-group of E with 3 elements. Since E has 9 elements, it cannot have any other non-zero proper subgroup of order other than 3. Thus G is the only prime Z-subgroup of E. Hence $\mathcal{A}$(E) = {Ann(G)}, which is singleton. So E is primary but not prime. //

2.3.10. Theorem [[15]] : Let E be with acc on annihilators then $\mathcal{A}$(E) = ∅ if and only if E = (0).

Proof : As the zero group has no prime subgroup, we have E = (0) ⇒ $\mathcal{A}$(E) = ∅.

Conversely, let E ≠ (0). Then the family $\mathcal{F}$ = {Ann(M) | M is a non-zero N-subgroup of E} ≠ ∅.

Since E satisfies the acc on annihilators, $\mathcal{F}$ contains a maximal element (say) Ann(M_1).

We claim that M_1 is a prime N-subgroup of E. For any non-zero N-subgroup M_2 of M_1, we get $Ann(M_2) \supseteq Ann(M_1)$. By maximality of $Ann(M_1)$, it follows that $Ann(M_1) = Ann(M_2)$. Hence M_1 is a prime N-subgroup of E. Thus $Ann(M_1) \in \mathcal{A}(E)$. In otherwords, $\mathcal{A}(E) \neq \emptyset$. Hence the result. //

2.3.11. <u>Definition</u> : Let E_1, E_2, E_3 be N-groups then the sequence $(0) \to E_1 \xrightarrow{f} E_2 \xrightarrow{g} E_3 \to (0)$ where f and g are N-homomorphims, is called an <u>exact sequence</u> if Im f = Ker g. Exactness of the sequence asserts that f is an N-monomorphism and g is an N-epimorphism.

2.3.12. <u>Theorem</u> [[15]] : Let E_1, E, E_2 be N-groups with an exact sequence $(0) \to E_1 \xrightarrow{f} E \xrightarrow{g} E_2 \to (0)$ then $\mathcal{A}(E_1) \subseteq \mathcal{A}(E) \subseteq \mathcal{A}(E_1) \cup \mathcal{A}(E_2)$.

<u>Proof</u> : Since f is injective and g is surjective, E = (0) gives $E_1 = (0) = E_2$ and thus the result follows.

Now, let $E \neq (0)$. The injectivity of f gives $f(E_1) \subseteq E$ and we can regard E_1 ($\cong f(E_1)$) as an N-subgroup of E.

If $Ann(M_1) \in \mathcal{A}(E_1)$, for some prime N-subgroup M_1 of E_1, then $Ann(M_1) \in \mathcal{A}(E)$ also, for M_1 is also a prime N-subgroup of E [by 2.3.6.] Thus, $\mathcal{A}(E_1) \subseteq \mathcal{A}(E)$.

Again, let $Ann(M) \in \mathcal{A}(E)$, for prime N-subgroup M of E. Then by 1.2.23, $M \cap E_1$ is an N-subgroup of E_1 as well as M (and hence of E).

Now, if $M \cap E_1 \neq (0)$, then M being a prime N-subgroup of E, we get $M \cap E_1$ is also a prime N-subgroup of E (by 2.3.6) and hence of E_1.

So, $\mathrm{Ann}(M) = \mathrm{Ann}(M \cap E_1) \in \mathcal{A}(E_1)$

Thus, $\mathcal{A}(E) \subseteq \mathcal{A}(E_1)$

Again, if $M \cap E_1 = (0)$ and $h = g|_M$ (the restriction of g to M) then for $m \in M$, $h(m) = 0$ implies $g(m) = 0$.

$\Rightarrow \quad m \in \mathrm{Ker}\, g = \mathrm{Im}\, f = f(E_1)$

$\Rightarrow \quad m \in M \cap f(E_1) \cong M \cap E_1 = (0)$

Thus, $m = 0$ which gives h is injective.

Hence $h(M) \cong M$.

So, M ($\subseteq E_2$) is a prime N-subgroup of E_2. Therefore, $\mathrm{Ann}(M) \in \mathcal{A}(E_2)$ which gives $\mathcal{A}(E) \subseteq \mathcal{A}(E_2)$ giving thereby $\mathcal{A}(E) \subseteq \mathcal{A}(E_1) \cup \mathcal{A}(E_2)$.

Thus, $\mathcal{A}(E_1) \subseteq \mathcal{A}(E) \subseteq \mathcal{A}(E_1) \cup \mathcal{A}(E_2)$. //

Now we give an example in which the equality between $\mathcal{A}(E)$ and $\mathcal{A}(E_1) \cup \mathcal{A}(E_2)$ fails.

2.3.13. Example [[15]] : If $E = Z$, the group of integers, $E_1 = 2Z$ and $E_2 = \mathbb{Z}/2\mathbb{Z}$ then $\mathcal{A}(E) \neq \mathcal{A}(E_1) \cup \mathcal{A}(E_2)$.

Now, if $M \cap E_1 \neq (0)$, then M being a prime N-subgroup of E, we get $M \cap E_1$ is also a prime N-subgroup of E (by 2.3.6) and hence of E_1.

So, $\text{Ann}(M) = \text{Ann}(M \cap E_1) \in \mathcal{A}(E_1)$

Thus, $\mathcal{A}(E) \subseteq \mathcal{A}(E_1)$

Again, if $M \cap E_1 = (0)$ and $h = g|_M$ (the restriction of g to M) then for $m \in M$, $h(m) = 0$ implies $g(m) = 0$.

$\Rightarrow \quad m \in \text{Ker } g = \text{Im } f = f(E_1)$

$\Rightarrow \quad m \in M \cap f(E_1) \cong M \cap E_1 = (0)$

Thus, $m = 0$ which gives h is injective.

Hence $h(M) \cong M$.

So, $M \; (\subseteq E_2)$ is a prime N-subgroup of E_2. Therefore, $\text{Ann}(M) \in \mathcal{A}(E_2)$ which gived $\mathcal{A}(E) \subseteq \mathcal{A}(E_2)$ giving thereby $\mathcal{A}(E) \subseteq \mathcal{A}(E_1) \cup \mathcal{A}(E_2)$.

Thus, $\mathcal{A}(E_1) \subseteq \mathcal{A}(E) \subseteq \mathcal{A}(E_1) \cup \mathcal{A}(E_2)$. //

Now we give an example in which the equality between $\mathcal{A}(E)$ and $\mathcal{A}(E_1) \cup \mathcal{A}(E_2)$ fails.

2.3.13. <u>Example</u> [[15]] : If $E = Z$, the group of integers, $E_1 = 2Z$ and $E_2 = Z/2Z$ then $\mathcal{A}(E) \neq \mathcal{A}(E_1) \cup \mathcal{A}(E_2)$.

<u>Proof</u> : $(Z,+)$, $(2Z,+)$ and $(Z/2Z, +)$ are Z-groups and we get an exact sequence

$$(0) \to 2Z \to Z \to Z/2Z \to (0)$$

But $Z/2Z = Z_2$ contains only two elements. So it has no non-zero proper subgroup. Therefore, Z_2 itself is the only prime Z-subgroup of it and $Ann(Z_2) = \langle 2 \rangle$ ($=2Z$).

Now, $\mathcal{A}(2Z) \cup \mathcal{A}(Z/2Z)$

$= \mathcal{A}(2Z) \cup \mathcal{A}(Z_2)$

$= (0) \cup 2Z = 2Z$

But $\mathcal{A}(Z) = 0$. Thus, $\mathcal{A}(Z) \neq \mathcal{A}(2Z) \cup \mathcal{A}(Z/2Z)$ //

2.3.14. <u>Theorem</u> [[15]] : If E_1 and E_2 are two N-groups then $\mathcal{A}(E_1 \oplus E_2) = \mathcal{A}(E_1) \cup \mathcal{A}(E_2)$

<u>Proof</u> : An arbitrary element of $\mathcal{A}(E_1 \oplus E_2)$ is $Ann(M)$, where M is a prime N-subgroup of $E_1 \oplus E_2$. Now, $M = M_1 \oplus M_2$, where M_1 and M_2 are N-subgroups of E_1 and E_2 respectively.

Since cach of M_1 and M_2 can be regarded as an N-subgroup of M, it follows that M_1 and M_2 are prime N-subgroups of M.

Therefore, $Ann(M_1) = Ann(M) = Ann(M_2)$.

Again, M_1 and M_2 are prime N-subgroups of E_1 and E_2 respectively also. It follows that

$Ann(M) \in \mathcal{A}(E_1), \mathcal{A}(E_2)$

Thus, $Ann(M) \in \mathcal{A}(E_1) \cup \mathcal{A}(E_2)$

So, $\mathcal{A}(E_1 \oplus E_2) \subseteq \mathcal{A}(E_1) \cup \mathcal{A}(E_2)$.

To get the opposite relation, let $M \in \mathcal{A}(E_1) \cup \mathcal{A}(E_2)$.

For definiteness, let $M \in \mathcal{A}(E_1)$. Then $M = Ann(S)$, for some prime N-subgroup S of E_1. Hence S is a prime N-subgroup of $E_1 \oplus E_2$ also. So $M \in \mathcal{A}(E_1 \oplus E_2)$.

Therefore, $\mathcal{A}(E_1) \cup \mathcal{A}(E_2) \subseteq \mathcal{A}(E_1 \oplus E_2)$ giving us the required equality. //

2.3.15. Theorem [[15]] : If M and Q are N-subgroups of E and T is an ideal of E such that $T \subseteq M, Q$ then

$$\mathcal{A}((M \cap Q)/_T) = \mathcal{A}(M/_T \cap Q/_T) \subseteq \mathcal{A}(M/_T) \cap \mathcal{A}(Q/_T)$$

Proof : Since M,Q are N-subgroups of E and so by 1.2.23, $M \cap Q$ is also an N-subgroup of E. Also, $T \subseteq M, Q$ gives $T \subseteq M \cap Q$.

Thus $M/_T$, $Q/_T$, $(M \cap Q)/_T$ are N-subgroups of $E/_T$ (by 1.2.39).

So $M/_T \cap Q/_T$ is also an N-subgroup of $E/_T$.

Again, $M \cap Q \subseteq M, Q$ and so $(M \cap Q)/_T \subseteq M/_T$, $Q/_T$,

Therefore, $(M \cap Q)/_T \subseteq M/_T \cap Q/_T$

Now, if $Ann(S/_T) \in \mathcal{A}((M \cap Q)/_T)$, for some prime N-subgroup

$S/_T$ of $(M \cap Q)/_T$ then $S/_T$ is also prime N-subgroup of $M/_T \cap Q/_T$ which gives $Ann(S/_T) \in \mathcal{A}(M/_T \cap Q/_T)$

Therefore $\mathcal{A}((M \cap Q)/_T) \subseteq \mathcal{A}(M/_T \cap Q/_T)$ ... (i)

Conversely, let $Ann(H/_T) \in \mathcal{A}(M/_T \cap Q/_T)$ for some prime N-group $H/_T$ of $M/_T \cap Q/_T$. Then $H/_T \subseteq M/_T, Q/_T$

$\Rightarrow H \subseteq M, Q$ ... (ii)

$\Rightarrow H \subseteq M \cap Q$

$\Rightarrow H/_T \subseteq (M \cap Q)/_T$

Hence $H/_T$ is a prime N-subgroup of $(M \cap Q)/_T$. Thus, $Ann(H/_T) \in \mathcal{A}(M \cap Q)/_T$ and so $\mathcal{A}(M/_T \cap Q/_T) \subseteq \mathcal{A}((M \cap Q)/_T)$

Therefore, (i) gives, $\mathcal{A}(M \cap Q)/_T = \mathcal{A}(M/_T \cap Q/_T)$.

Again (ii) implies that $H/_T$ is a prime N-subgroup of $M/_T$ and $Q/_T$. Thus, $Ann(H/_T) \in \mathcal{A}(M/_T)$ and $Ann(H/_T) \in \mathcal{A}(Q/_T)$. So $Ann(H/_T) \in \mathcal{A}(M/_T) \cap \mathcal{A}(Q/_T)$.

Therefore, $\mathcal{A}(M/_T \cap Q/_T) \subseteq \mathcal{A}(M/_T) \cap \mathcal{A}(Q/_T)$. //

2.4. N-groups with finite spanning dimension

It is seen that Fleury [18], Rangaswami [43] and Satyanarayana [44, 46] have studied some properties of modules with finite spanning dimension (fsd). The authors of [8] and [12] have also studied N-groups with finite spanning dimension. We define fsd here in a different way.

2.4.1. <u>Definitions</u> : An ideal J of an N-group E is said to be an <u>ideal with fsd-1</u> if for every strictly descending sequence $J \supset J_1 \supset J_2 \supset \ldots$ of ideals of E there exists an $i \in Z^+$ such that $J_j \subset_s J$ for all $j \geq i$.

An N-group E is called an <u>fsd-1 N-group</u> if E is with fsd-1 as an ideal.

It is clear that if every ideal of E is with fsd-1 then E is an fsd-1 N-group.

2.4.2. <u>Proposition</u> [[15]] : If E is an fsd-1 N-group and H is an ideal of E then $E/_H$ is with fsd-1.

<u>Proof</u> : Let $E/_H \supset E_1/_H \supset E_2/_H \supset \ldots$ (i)

be a strictly descending chain of ideals of $E/_H$. Here $E_1, E_2, E_3, \ldots$ are ideals of E and each one containing H.

Thus, the chain (i) gives us another strictly descending chain $E \supset E_1 \supset E_2 \supset \ldots$ of ideals of E.

Since E is fsd-1 N-group , there exists a $j \in Z^+$ such that $E_i \subset_s E$ for all $i \geq j$.

Now, if possible, let for all i, $E_i/_H \not\subset_s E/_H$. Thus, by

2.1.4(a), $E_i \not\subset_s E$ for all i. This gives a contradiction with $E_i \subset_s E$ for $i \geq j$. Hence our supposition is wrong and so there exists at least one $E_i/_H$ in (i) such that $E_i/_H \subset_s E/_H$. And then $E/_H$ is with fsd-1. //

2.4.3. Definition : An ideal E_1 and an N-group E is said to be hollow if every ideal L ($\subset E_1$) of E is small in E_1.

It follows immediately from 2.1.4 that

2.4.4. Proposition : If L is a hollow ideal of E then all the ideals of E properly contained in L are small in E. //

The definition of a supplement here is an extension of the definition due to Satyanarayana [44].

2.4.5. Definitions : An ideal M of E is said to be a supplement of an ideal M' of E if $M + M' = E$, $L + M' \neq E$ for any ideal L ($\subset M$) of E.

By a supplement M in E we shall mean M is a supplement of some ideal M_1 of E in E.

2.4.6. Proposition : If L_o and L are ideals of E such that $L_o \not\subset_s L$ and $T \subset_s L$, for any ideal $T(\subset L_o)$ of E then L_o is hollow.

Proof : Let L_o be not hollow. Then there exists an ideal A of E with $A \subset L_o$ such that $A \not\leq_s L_o$. Thus we get an ideal B of E with $B \subset L_o$ such that $A + B = L_o$ . . . (i)

Again, since $L_o \not\leq_s L$ then there exists an ideal X_o ($\subset L$) of E such that

$$L_o + X_o = L .$$

$$\Rightarrow \quad A + B + X_o = L, \text{ [by (i)]} \quad \ldots \quad \text{(ii)}$$

Now, since A is an ideal of E with $A \subset L_o$ and so $A \subset_s L$ by our hypothesis. Thus (ii) gives $B + X_o = L$. Again, by the similar argument, we have $B \subset_s L$. Hence $X_o = L$, a contradiction. Therefore L_o is hollow. //

This result propels us to the following.

2.4.7. Remark : If L_o and L are ideals of E such that $L_o \not\leq_s L$ and L_o is not hollow then there exists at least one ideal L_1 ($\subset L_o$) of E such that $L_1 \not\leq_s L$.

2.4.8. Proposition : If an ideal L of E is with fsd-1 and L_o($\subseteq L$) is an ideal of E such that $L_o \not\leq_s L$ then L_o contains a hollow ideal H of E with $H \not\leq_s L$.

Proof : If L_o is hollow then we are done. If L_o is not hollow then by 2.4.7 we have an ideal L_1($\subset L_o$) of E such that $L_1 \not\leq_s L$.

of E with $A \subset L_0$ such that $A \not\leq_s L_0$. Thus we get an ideal B of E with $B \subset L_0$ such that $A + B = L_0$ ... (i)

Again, since $L_0 \not\leq_s L$ then there exists an ideal X_0 ($\subset$ L) of E such that

$$L_0 + X_0 = L .$$

$$\Rightarrow \quad A + B + X_0 = L, \text{ [by (i)]} \quad \ldots \quad \text{(ii)}$$

Now, since A is an ideal of E with $A \subset L_0$ and so $A \subset_s L$ by our hypothesis. Thus (ii) gives $B + X_0 = L$. Again, by the similar argument, we have $B \subset_s L$. Hence $X_0 = L$, a contradiction. Therefore L_0 is hollow. //

This result propels us to the following.

2.4.7. <u>Remark</u> : If L_0 and L are ideals of E such that $L_0 \not\leq_s L$ and L_0 is not hollow then there exists at least one ideal L_1 ($\subset L_0$) of E such that $L_1 \not\leq_s L$.

2.4.8. <u>Proposition</u> : If an ideal L of E is with fsd-1 and $L_0 (\subseteq L)$ is an ideal of E such that $L_0 \not\leq_s L$ then L_0 contains a hollow ideal H of E with $H \not\leq_s L$.

<u>Proof</u> : If L_0 is hollow then we are done. If L_0 is not hollow then by 2.4.7 we have an ideal $L_1(\subset L_0)$ of E such that $L_1 \not\leq_s L$.

Now, if L_1 is hollow then we are done. If not, we get a strictly descending chain $L_o \supset L_1 \supset L_2 \supset \ldots$, where each ideal $L_i \not\leq_s L$.

But L being with fsd-1, there exists an ideal L_j such that $L_\alpha (\subset L_j)$ is small in L for $\alpha \geq j$. And this is a contradiction.

So, one of the L_i's (say H) must be hollow and is such that $H \not\leq_s L$. //

As a corollary to the above result we get

2.4.9. <u>Corollary</u> [[15]] : If E is an fsd-1 N-group then every non small ideal of it contains a hollow ideal H with $H \not\leq_s E$.

<u>Proof</u> : Considering E as an ideal of itself, the result follows immediately from 2.4.8. //

2.4.10 . <u>Proposition</u> : If an ideal L of E is with fsd-1 then every ideal M ($\subseteq$ L) of E has a supplement in L.

<u>Proof</u> : Case (i) :

Let M = L then L + (0) = L and clearly (0) is the supplement of L in L.

Case (ii) :

Let $M \neq L$. Now, if $M \subset_s L$ then for any ideal X ($\subseteq$ L) of E with M + X = L we get X = L.

So, for any ideal X_1 ($\subset X$) of E, $M + X_1 \neq L$. Otherwise, $M + X_1 = L \Rightarrow X_1 = L = X$. Thus L is a supplement of M in L.

Next, if $M \not\subset_s L$, then there is an ideal X_o ($\subset L$) of E such that $M + X_o = L$.

Now, if X_o is a supplement of M in L then we are done. If not, we get an ideal X_1 ($\subset X_o$) of E such that $M + X_1 = L$. If $X_1 \subset_s L$ then $M = L$, a contradiction. Therefore, $X_1 \not\subset_s L$.

If X_1 is a supplement of M in L then we are done. If not, we get an ideal X_2 ($\subset X_1$) of E such that $M + X_2 = L$. By the similar argument as above we have $X_2 \not\subset_s L$.

Proceeding in this way, we get a strictly descending chain of ideals of E.

$$L \supset X_o \supset X_1 \supset X_2 \supset \dots$$

where X_i is not a supplement of M in L and $X_{i+1} \not\subset_s L$ $(i=0,1,2,\dots)$

Since L is with fsd-1, we get a $t \in Z^+$ such that $X_\alpha \subset_s L$ for all $\alpha \geq t$ and this is clearly a contradiction. So the process must come to an end after a finite steps. In otherwards, we must have some $t \in Z^+$ such that X_t($\subset L$) is a supplement of M in L. //

2.4.11. Corollary [[15]] : If E is an fsd-1 N-group then every ideal of it has a supplement.

Proof : Considering E as the ideal L in 2.4.10, the result follows immediately. //

2.4.12. <u>Proposition</u> : If E is an fsd-1 N-group and L,H are two ideals of E such that L($\neq$ E) is a supplement of H in E then H contains a supplement of L in E.

<u>Proof</u> : Since L is a supplement of H in E then L + H = E and L_1 + H $\neq$ E for any ideal L_1($\subset$ L) of E.

Now H $\subset_s$ E $\Rightarrow$ L = E, a contradiction.

Therefore H $\not\subset_s$ E.

If H is a supplement of L in E then we are done. If not, then we have an ideal H_1($\subset$ H) of E such that L + H_1 = E. Clearly as above, $H_1 \not\subset_s$ E.

Again if H_1 is a supplement of L in E then we are done. If not, we get an ideal H_2 ($\subset H_1$) of E such that L + H_2 = E and hence $H_2 \not\subset_s$ E.

Proceeding in this way we get a strictly descending chain of ideals of E ,

$$H \supset H_1 \supset H_2 \supset \cdots \quad \text{where } H, H_i \not\subset_s E \text{ (for all i)}$$

This contradicts the fsd-1 character of E. So there must exist an ideal H_i ($\subset$ H) of E which is a supplement of L in E. //

2.4.13. <u>Corollary</u> [[15]] : If E is an fsd-1 N-group then every non small ideal of E contains a non-zero supplement in E.

Proof : Let I be a non small ideal of E. Then by 2.4.11, we get an ideal M(say) of E such that M is a supplement of I in E.

So, $M + I = E$ and $M_1 + I \neq E$ for any ideal M_1 ($\subset M$) of E. Thus by 2.4.12, I contains an ideal I_1 of E such that I_1 is a supplement of M in E.

Therefore, $M + I_1 = E$ and $M + I_2 \neq E$, for any ideal I_2($\subset I_1$) of E.

Now if $I_1 = (0)$ then $M = E$. Thus $M + I = E$ implies $I \subset_s E$, a contradiction. Therefore, a non small ideal I contains a non-zero supplement I_1 of M in E. //

2.4.14. Proposition : If an ideal L of E is with fsd-1 then there is a $t \in Z^+$ such that $L = \sum_{i=1}^{t} L_i$ where each ideal L_i ($\subseteq L$) of E is hollow and $\sum_{i \neq j}^{t} L_i \neq L$, for $1 \leq i, j \leq t$.

Proof : We know that $L \not\subseteq_s L$ and hence by 2.4.8, L contains a hollow ideal L_1(say) of E with $L_1 \not\subseteq_s L$.

Now, if $L_1 = L$ then we are done. If not, let $L_1 \subset L$ then by 2.4.10, L_1 has a supplement X_1 in L where X_1 ($\subset L$) is an ideal of E.

So, $L_1 + X_1 = L$ ... (i)

and $L_1 + Y_1 \neq L$ for any ideal Y_1($\subset X_1$) of E.

If $X_1 \subset_s L$ then $L_1 = L$ [from (i)], a contradiction. So $X_1 \not\subset_s L$. Hence by 2.4.8, X_1 contains a hollow ideal L_2 (say) with $L_2 \not\subset_s L$.

Now, if $L_2 \subset_s X_1$ then by 2.1.5, $L_2 \subset_s L$, a contradiction. Thus $L_2 \not\subset_s X_1$. Therefore, there exists an ideal X_2 ($\subset X_1$) of E such that

$$L_2 + X_2 = X_1 \qquad \cdots \qquad \text{(ii)}$$

Deletion of any one of L_2 and X_2 will give a proper subset of X_1. Thus no deletion is possible in (ii).

Again (i) and (ii) give us

$$L_1 + L_2 + X_2 = L.$$

Now, if $X_2 \subset_s L$ then $L_1 + L_2 = L$ and thus the result follows. And if $X_2 \not\subset_s L$ then by 2.4.8, X_2 contains a hollow ideal L_3 (say) of E such that $L_3 \not\subset_s L$. Repeating the process as above, we get $L = L_1 + L_2 + L_3 + \cdots$ and a chain $L \supseteq X_1 \supset X_2 \supset X_3 \supset \cdots$ where each ideal $X_i \not\subset_s L$ and each L_i is hollow. Here we meet a contradiction with fsd-1 character of L. So we must have some $m \in Z^+$ such that $X_{m+1} \subset_s L$.

And as we have discussed above,

$$L = L_1 + L_2 + \cdots + L_{m+1} + X_{m+1} \text{ where each } L_i \text{ is hollow.}$$

Thus using $X_{m+1} \subset_s L$ we get

If $X_1 \subset_s L$ then $L_1 = L$ [from (i)], a contradiction. So $X_1 \not\subset_s L$. Hence by 2.4.8, X_1 contains a hollow ideal L_2 (say) with $L_2 \not\subset_s L$.

Now, if $L_2 \subset_s X_1$ then by 2.1.5, $L_2 \subset_s L$, a contradiction. Thus $L_2 \not\subset_s X_1$. Therefore, there exists an ideal X_2 ($\subset X_1$) of E such that

$$L_2 + X_2 = X_1 \qquad \ldots \qquad \text{(ii)}$$

Deletion of any one of L_2 and X_2 will give a proper subset of X_1. Thus no deletion is possible in (ii).

Again (i) and (ii) give us

$$L_1 + L_2 + X_2 = L.$$

Now, if $X_2 \subset_s L$ then $L_1 + L_2 = L$ and thus the result follows. And if $X_2 \not\subset_s L$ then by 2.4.8, X_2 contains a hollow ideal L_3 (say) of E such that $L_3 \not\subset_s L$. Repeating the process as above, we get $L = L_1 + L_2 + L_3 + \ldots$ and a chain $L \supseteq X_1 \supset X_2 \supset X_3 \supset \ldots$ where each ideal $X_i \not\subset_s L$ and each L_i is hollow. Here we meet a contradiction with fsd-1 character of L. So we must have some $m \in Z^+$ such that $X_{m+1} \subset_s L$.

And as we have discussed above,

$$L = L_1 + L_2 + \ldots + L_{m+1} + X_{m+1} \text{ where each } L_i \text{ is hollow.}$$

Thus using $X_{m+1} \subset_s L$ we get

$$L = L_1 + L_2 + \dots + L_{m+1}$$

So, $L = \sum_{i=1}^{t} L_i$, $t = m+1 \in Z^+$

Deletion of any term from the right hand side gives rise a proper subset of L. Hence no term can be deleted and so $\sum_{i \neq j}^{t} L_i \neq L$ for $1 \leq j \leq t$. //

2.4.15. Corollary [[15]] : If E is an fsd-1 N-group then there is a $t \in Z^+$ such that $E = \sum_{i=1}^{t} E_i$, where each E_i is hollow ideal and $\sum_{i \neq j}^{t} E_i \neq E$ for $1 \leq i, j \leq t$.

Proof : Considering E as the ideal L in 2.4.14, the result follows immediately. //

2.4.16. Proposition : If in the above proposition 2.4.14, $L = \sum_{i=1}^{r} L_i'$, where each ideal L_i' ($\subseteq L$) of E is hollow and $\sum_{i \neq j}^{r} L_i' \neq L$ for $1 \leq i,j \leq r$, then $t = r$.

Proof : Let $r > t$. We first show that for some $i(1 \leq i \leq r)$, $L_i' + L_2 + L_3 + \dots + L_t = L$ and none of the terms in the sum can be deleted.

Suppose that $L_1' + L_2 + L_3 + \ldots + L_t \neq L$ ... (i)

Then $L_1' + L_2 + L_3 + \ldots + L_t \neq L_1 + L_2 + \ldots + L_t (= L)$.

Hence for some ideal U ($\subset L_1$) of E, we get

$L_1' + L_2 + L_3 + \ldots + L_t = U + L_2 + L_3 + \ldots + L_t$ ... (ii)

Now, $L_2' + L_3' + \ldots + L_r' + U + L_2 + L_3 + \ldots + L_t$

$= L_2' + L_3' + \ldots + L_r' + L_1' + L_2 + L_3 + \ldots + L_t$ [by(ii)]

$= L_1' + L_2' + L_3' + \ldots + L_r' + L_2 + L_3 + \ldots + L_t$

$= L + L_2 + L_3 + \ldots + L_t$ (as $L = L_1' + \ldots + L_r'$)

$= L$, since each $L_i \subseteq L$

Therefore, if (i) holds then we get

$L_2' + L_3' + \ldots + L_r' + L_2 + L_3 + \ldots + L_t + U = L$ ... (iii)

And each L_i being hollow and $U \subset L_1$, so $U \subset_s L_1$. Thus by 2.1.5, $U \subset_s L$ which together with (iii) give

$L_2' + L_3' + \ldots + L_r' + L_2 + L_3 + \ldots + L_t = L$ ... (iv)

Again, suppose $L_2' + L_2 + L_3 + \ldots + L_t \neq L$ ... (v)

Then for some ideal V ($\subset L_1$) of E, we have

$L_2' + L_2 + L_3 + \ldots + L_t = V + L_2 + L_3 + \ldots + L_t$ (as $L = \sum_{i=1}^{t} L_i$)

Also, $L_3' + L_4' + \ldots + L_r' + V + L_2 + L_3 + \ldots + L_t$

$= L_3' + L_4' + \dots + L_r' + L_2' + L_2 + \dots + L_t$

$= L_2' + L_3' + L_4' + \dots + L_r' + L_2 + L_3 + \dots + L_t$

$= L$ [by (iv)]

Hence, if (i) and (v) hold, then we get

$$L_3' + L_4' + \dots + L_r' + L_2 + L_3 + \dots + L_t + V = L.$$

Now, $V \subset_s L$ gives

$$L_3' + L_4' + \dots + L_r' + L_2 + L_3 + \dots + L_t = L .$$

If we continue the process, we finally get

$$L_r' + L_2 + L_3 + \dots + L_t = L \quad \text{and}$$

$$L_j' + L_2 + L_3 + \dots + L_t \neq L \text{ for all } j = 1,2,\dots, r-1 .$$

Thus for $1 \leq j \leq r-1$, $L_j' + L_2 + \dots + L_t \neq L$ implies $L_r' + L_2 + L_3 + \dots + L_t = L$.

Hence there exists an $i \leq r$ such that

$$L_i' + L_2 + L_3 + \dots + L_t = L \qquad \dots \qquad \text{(vi)}$$

Now, if L_i' is deleted from the left side of (vi) then $L_2 + L_3 + \dots + L_t = L$, a contradiction. If L_2 is deleted then (vi) becomes $L_i' + L_3 + L_4 + \dots + L_t = L$. But $L_2 + L_3 + \dots + L_t \neq L$. Hence $L_2 + L_3 + \dots + L_t = W + L_3 + L_4 + \dots + L_t$ where W is an ideal ($\subset L_i'$) of E.

As L_i' is hollow and $W \subset L_i'$, it follows that $W \subset_s L_i'$ and hence $W \subset_s L$.

Therefore, $L_1 + W + L_3 + L_4 + \ldots + L_t = L_1 + L_2 + \ldots + L_t = L$.

$$\Rightarrow \quad L_1 + L_3 + L_4 + \ldots + L_t = L$$

This is not true. Hence L_2 cannot be deleted. Similarly we see that no term of (vi) can be deleted.

Again we are given that $L_1 + L_2 + \ldots + L_t = L$ .. (vii)

Comparing this with (vi), it follows that L_1 is replaced by some $L_i' \in \{ L_1', L_2', \ldots, L_r' \}$. Similarly, each of $L_2, L_3, \ldots, L_t$ can be replaced by some other members of $\{L_1', L_2', \ldots, L_r'\}$.

Now for simplicity, let us consider that all the terms of (vii) $L_1, L_2, \ldots, L_t$ are replaced by $L_1', L_2', \ldots, L_t'$ respectively. Then we get from (vii), $L_1' + L_2' + \ldots + L_t' = L$ and no term can be deleted from the sum. But $L_1' + L_2' + \ldots + L_r' = L$ and no term can be deleted from this sum. This gives a contradiction. So $r \nless t$ and similarly $t \nless r$. Hence $t = r$. //

2.4.17. <u>Definition</u> : If an ideal L of E is with fsd-1 such that $L = \sum_{i=1}^{t} L_i$ where each ideal $L_i (\subset L)$ of E is hollow and no term of the sum can be deleted then the definite positive integer t is called <u>the spanning dimension of L</u> and is denoted by $Sd_1 (L) = t$.

As a corollary to the proposition 2.4.16, we get

2.4.18. Corollary [[15]] : If E is an fsd-1 N-group then there exists a $t \in Z^+$ such that $Sd_1(E) = t$.

Proof : By 2.4.15, we get a $t \in Z^+$ such that $E = \sum_{i=1}^{t} E_i$ with each ideal E_i hollow and $E \neq \sum_{i \neq j}^{t} E_i$ for $1 \leq i, j \leq t$.

Now from 2.4.16, it follows that such a t is unique. Hence $Sd_1(E) = t$. //

2.4.19. Proposition : If E is an fsd-1 N-group and an ideal M of E is a supplement in E then M is with fsd-1.

Proof : Let us consider a strictly descending chain of ideals of E such that

$$M \supset X_1 \supset X_2 \supset X_3 \supset \ldots \qquad \text{(i)}$$

where M is a supplement of an ideal H of E.

Then $M + H = E$ ... (ii)

and $M_1 + H \neq E$, where M_1 ($\subset M$) is an ideal of E.

Suppose $X_i \not<_s M$ for each i. Then there exists an ideal $A (\subset M)$ of E such that $X_i + A = M$ ... (iii)

Since M is a supplement of H in E,

$$A + H \neq E \qquad \ldots \qquad \text{(iv)}$$

Now (ii) gives

$$(X_i + A) + H = E \quad \text{[by (iii)]}$$

$$\Rightarrow \quad X_i + (A + H) = E$$

$$\Rightarrow \quad X_i \not\subset_s E \quad \text{(because of (iv))}$$

Thus $X_i \not\subset_s M \Rightarrow X_i \not\subset_s E$ for each i.

And this contradicts that E is an fsd-1 N-group. Hence we must have some j such that $X_\alpha \subset_s M$ for all $\alpha \geq j$. In otherwords, M is with fsd-1. //

2.4.20. Proposition [[15]] : If E is an fsd-1 N-group and L, M are ideals of E where each of them is a supplement of the other in E then

$$Sd_1(E) = Sd_1(L) + Sd_1(M)$$

Hence, if $Sd_1(E) = Sd_1(M)$ then E = M.

Proof : Since L and M are supplements to each other in E and E is an fsd-1 N-group then by 2.4.19, each of L and M is with fsd-1. And let $Sd_1(L) = t$ and $Sd_1(M) = r$ [by 2.4.18].

Now, by 2.4.14, we can write

$$L = \sum_{i=1}^{t} L_i,$$ where each ideal $L_i (\subset L)$ of E is hollow

and $M = \sum_{j=1}^{r} M_j$, where each $M_j (\subset M)$ of E is hollow and no term of the sums can be deleted.

Again, $E = L + M$ as L and M are supplements to each other and so no L or M can be deleted.

Therefore, $E = \sum_{i=1}^{t} L_i + \sum_{j=1}^{r} M_j$ and deletion of any L_i or M_j is not permissible. Hence E is the sum of t numbers of hollow ideals L_i and r numbers of hollow ideals M_j and this sum is minimal as no L_i or M_j can be deleted.

Thus, $Sd_1(E) = t+r = Sd_1(L) + Sd_1(M)$.

For the next part, if $Sd_1(L) > 0$ then $Sd_1(E) > Sd_1(M)$, a contradiction with $Sd_1(E) = Sd_1(M)$.

Therefore, $Sd_1(L) = 0$ and so $L = (0)$.

Hence $E = M$ as $E = L + M$. /

2.4.21. <u>Corollary</u> [[15]] : If E is an fsd-1 N-group then E satisfies the acc on supplements in E.

<u>Proof</u> : Let $E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots$ be an ascending chain of supplements in E.

Here we note that each of E_i is with fsd-1 [by 2.4.19].

Since $E_1 \subseteq E_2$ and E_2 is with fsd-1, we get by 2.4.10, E_1 has a supplement B_1(say) in E_2 .

Thus $E_1 + B_1 = E_2$.

Again, by 2.4.19, B_1 is with fsd-1 .

Now from 2.4.14, let $E_1 = \sum_{i=1}^{r} H_i$, $B_1 = \sum_{j=1}^{t} T_j$ where each ideal H_i and T_j of E is hollow. Also let $E_2 = \sum_{k=1}^{p} L_k$ where each ideal L_k of E is hollow with $Sd_1(E_1) = r$, $Sd_1(B_1) = t$ and $Sd_1(E_2) = p$. Then

$$\sum_{i=1}^{r} H_i + \sum_{j=1}^{t} T_j = \sum_{k=1}^{p} L_k$$

$$\Rightarrow \quad r + t = p$$

$$\Rightarrow \quad r \leq p$$

$$\Rightarrow \quad Sd_1(E_1) \leq Sd_1 (E_2).$$

Therefore, from the considered chain, we get

$$Sd_1(E_1) \leq Sd_1(E_2) \leq Sd_1(E_3) \leq \ldots \quad [\leq Sd_1(E)].$$

But $Sd_1(E)$ is a definite number and hence for some $\alpha \in Z^+$, $Sd_1(E_\alpha) = Sd_1(E_{\alpha+1}) = \ldots = Sd_1(E)$.

Hence by 2.4.20, $E_\alpha = E_{\alpha+1} = \ldots = E$. Therefore the chain of supplements must be finite one. So E satisfies the acc on supplements in E. //

2.4.22 Proposition [[15]] : If E is an fsd-1 N-group such that the sum of any two supplements is again a supplement, M is a supplement in E, $\pi : E \to E/_M$ is a canonical mapping and $A/_M$ is a supplement in $E/_M$ then $\pi^{-1}(A/_M)$ is a supplement in E.

<u>Proof</u> : Let $A/_M$ be a supplement of $B/_M$ in $E/_M$.

Then $A/_M + B/_M = E/_M$ ... (i)

$\Rightarrow \quad (A+B)/_M = E/_M$

$\Rightarrow \quad A + B = E$

This implies that A contains a supplement A_1(say) of B in E. Thus, $A_1 + B = E$.

Now, if $(A_1+ M)/_M \subset A/_M$ then $(A_1+M)/_M + B/_M \neq E/_M$ as $A/_M$ is a supplement of $B/_M$.

But $(A_1+ M)/_M + B/_M = (A_1+ M + B)/_M = (E + M)/_M$ (as $A_1+B=E$) $= E/_M$, a contradiction.

Therefore, $(A_1+ M)/_M = A/_M$

$\Rightarrow \quad A_1 + M = A$

Thus $\pi^{-1}\left(A/_M\right) = A = A_1 + M$.

Since A_1 and M are supplements in E, so $A_1 + M$ is also a supplement in E by our hypothesis. Hence $\pi^{-1}(A/_M)$ is a supplement in E. //

2.4.23. Proposition : If E is an fad-1 N-group and H is a non-zero supplement in E then $E/_H$ satisfies the dcc on ideals of $E/_H$.

Proof : Let a strictly descending chain of ideals of $E/_H$ be

$$E/_H \supset E_1/_H \supset E_2/_H \cdots \qquad \text{(i)}$$

where each E_i is an ideal of E such that $E_i \supseteq H$.

If possible, let the chain be infinite and correspondingly we get a strictly descending infinite chain of ideals of E as $E \supset E_1 \supset E_2 \supset \cdots$

Since E is an fsd-1 N-group, there exists a $j \in Z^+$ such that $E_i \subset_s E$ for all $i \geq j$.

But $H \subseteq E_i$ for all i, hence by 2.1.4(a), $H \subset_s E$. Let H be a supplement of A in E then $H + A = E$ which gives $A = E$ as $H \subset_s E$. Therefore H is a supplement of E in E. But E has unique supplement (0). Hence $H = (0)$, a contradiction.

Therefore, the chain (i) must be a finite one and so it stops after a finite steps. Hence $E/_H$ satisfies the dcc on ideals of $E/_H$. //

2.5. s-p decomposition of (0) in an fsd-1 N-group

In a unique factorisation domain, one can express a non unit as a finite product $p_1^{\alpha_1}\, p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ of positive powers of distinct primes. This result can be expressed in terms of ideals as

$$\langle a\rangle = \langle p_1^{\alpha_1}\rangle \cap \langle p_2^{\alpha_2}\rangle \cap \ldots \cap \langle p_t^{\alpha_t}\rangle$$

A similar decomposition of ideals of a commutative Noetherian ring is known. We extend some results analogous to this theory to fsd-1 N-groups.

2.5.1. Definition [[15]] : For a prime N-subgroup H_1 of E, if there is an ideal H of E which is a prime N-subgroup of E and $H_1 \subseteq H$ then H is called a prime ideal extension of a prime N-subgroup H_1.

Also, a left ideal near-ring N in which every left N-subgroup is a left ideal is such an example where any prime N-subgroup clearly possesses a prime ideal extension.

2.5.2. Example (E(13), Page 339-340, [42]) :

N = {0,a,b,c} is a near-ring under addition [defined in table 1.1.(i)] and multiplication defined by the following table

.	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

Table : 2.3 .

Here $\{0,a\}$, $\{0,b\}$, $\{0,c\}$ and N are non-zero left N-subgroups of N and each of them is a left ideal of N.

Annihilator of each of the above N-subgroups of ${}_N N$ is $\{0,b\}$ and hence each of them is a prime N-subgroup of ${}_N N$. Moreover, each of them has prime ideal extension ${}_N N$. //

2.5.3. <u>Definition</u> [[15]] : An N-group E is said to be <u>prime complete</u> if for any ideal H of E the following condition holds : H^* is a prime N-subgroup of $E/_H$ implies H^* has a prime ideal extension in $E/_H$.

The following example will show the existence of such N-groups.

2.5.4. <u>Example</u> (Clay [16], 1968, 2.1.10) :

N = $\{0,a,b,c\}$ is a near-ring under addition and multiplication defined by the following tables.

+	0	a	b	c
0	0	a	b	c
a	a	b	c	0
b	b	c	0	a
c	c	0	a	b

(i)

.	0	a	b	c
0	0	0	0	0
a	0	a	b	a
b	0	b	0	b
c	0	c	b	c

(ii)

Table : 2.4.

Here N-group $_NN$ has only one non-trivial N-subgroup $\{0,b\}=L$ (say). This L is also an ideal of $_NN$. Thus L is the prime ideal extension of prime N-subgroup L itself.

Again, clearly $N/_L = \{\bar{0},\bar{a}\}$ and it is a prime N-group and so obviously it is prime complete. //

It is observed that there are some near-ring groups in which the sum of any two supplements is again a supplement. Besides the trivial cases of N-group E having no other ideal except (0) and E, we give the following example.

2.5.5. <u>Example</u>. In the example 2.5.2., we observe that $\{0,a\}$, $\{0,b\}$, $\{0,c\}$ are non trivial ideals of $_NN$ and each of them is a supplement in $_NN$. It can be verified easily that the sum of any two of them gives the N-group $_NN$ which is also a supplement.

Moreover this example is sufficient to say that supplement of an ideal need not be unique. For, $\{0,b\}$ and $\{0,c\}$ both are supplements of $\{0,a\}$ in $_NN$. //

Now, in what follows (as in [[15]]), we confine our discussion on prime complete E where a prime N-subgroup of its factors has non-small prime ideal extension (shortly, <u>a prime complete E with non-small extension</u>) and sum of any two supplements is again a supplement.

2.5.6. <u>Theorem</u> [[15]] : Let E be an fsd-1 N-group with acc on annihilators. If $\mathcal{A}(E) = X \cup Y$ where $X \cap Y = \emptyset$ then there exists a supplement E' in E such that $\mathcal{A}(E') = X$ and $\mathcal{A}(E/_{E'}) = Y$.

Proof : Let $\mathcal{H}$ be the collection of supplements H in E such that $\mathcal{A}(H) \subseteq X$.

Here $\mathcal{H} \neq \phi$ for $\mathcal{A}(0) = \emptyset \subseteq X$.

Now, since E is an fsd-1 N-group, by 2.4.21, E satisfies the acc on supplements in E and hence $\mathcal{H}$ has a maximal element E'.

Therefore $\mathcal{A}(E') \subseteq X$ ... (i)

Again from the exact sequence

$$(0) \to E' \to E \to E/_{E'} \to (0)$$

as in 2.3.12, we get

$$\mathcal{A}(E) \subseteq \mathcal{A}(E') \cup \mathcal{A}(E/_{E'}) \quad \ldots \quad \text{(ii)}$$

And this gives

$$X \cup Y \subseteq \mathcal{A}(E') \cup \mathcal{A}(E/_{E'}) \subseteq X \cup \mathcal{A}(E/_{E'}) \quad , \quad \text{[by (i)]}$$

Now, since $X \cap Y = \emptyset$, it follows from the above, $Y \subseteq \mathcal{A}(E/_{E'})$ ...(iii)

Suppose, $Y \subset \mathcal{A}(E/_{E'})$. Then there exists a prime N-subgroup $H'/_{E'}$ of $E/_{E'}$ such that

$$\text{Ann}(H'/_{E'}) \in \mathcal{A}(E/_{E'}) \text{ and } \text{Ann}(H'/_{E'}) \notin Y \quad \ldots \quad \text{(iv)}$$

Since E is prime complete with non-small extension and $H'/_{E'}$ is a prime N-subgroup of $E/_{E'}$, $H'/_{E'}$ has a non-small prime ideal extension

H/E' (say). By 2.4.2, E/E' is an fsd-1 N-group and so [by 2.4.13] H/E' contains a non-zero supplement T/E' in E/E'.

Therefore, 2.4.22 reveals that T is a supplement in E. Also, $\text{Ann}(T/E') = \text{Ann}(H/E')$ as H/E' is prime and so T/E' ($\subseteq H/E'$) is also prime. Thus $\mathcal{A}(T/E')$ is singleton.

So, let $\mathcal{A}(T/E') = \{P\}$ ... (v)

Moreover, $\text{Ann}(H/E') = \text{Ann}(T/E') = P$ ... (vi)

Also, since $H'/E' \subseteq H/E'$ and H/E' is prime,

$\text{Ann}(H'/E') = \text{Ann}(H/E') = P$, [by (vi)]

Therefore (iv) gives $P \not\subseteq Y$ ... (vii)

Again from the exact sequence

$$(0) \to E' \to T \to T/E' \to (0)$$

(where $E' \subset T$) we have, from 2.3.12, $\mathcal{A}(T) \subseteq \mathcal{A}(E') \cup \mathcal{A}(T/E')$

So $\mathcal{A}(T) \subseteq X \cup \{P\}$, [by (i) and (v)] ... (viii)

Since $T \subseteq E$, by the definition, we get $\mathcal{A}(T) \subseteq \mathcal{A}(E)$.

Thus $\mathcal{A}(T) \subseteq X \cup Y$ ... (ix)

Therefore, $\mathcal{A}(T) \subseteq (X \cup \{P\}) \cap (X \cup Y)$, [by (viii) and (ix)]

$\Rightarrow \mathcal{A}(T) \subseteq (X \cap (X \cup Y)) \cup (\{P\} \cap (X \cup Y))$

$\Rightarrow \mathcal{A}(T) \subseteq X \cup (\{P\} \cap (X \cup Y))$

Now, if $P \not\subseteq X$ then $\mathcal{A}(T) \subseteq X \cup \emptyset$, [by (vii)]

H/E' (say). By 2.4.2, E/E' is an fsd-1 N-group and so [by 2.4.13] H/E' contains a non-zero supplement T/E' in E/E'.

Therefore, 2.4.22 reveals that T is a supplement in E. Also, Ann (T/E') = Ann (H/E') as H/E' is prime and so T/E' $(\subseteq H/E')$ is also prime. Thus $\mathcal{A}(T/E')$ is singleton.

So, let $\mathcal{A}(T/E') = \{P\}$ ... (v)

Moreover, Ann (H/E') = Ann $(T/E') = P$ ... (vi)

Also, since $H'/E' \subseteq H/E'$ and H/E' is prime,

Ann (H'/E') = Ann $(H/E') = P$, [by (vi)]

Therefore (iv) gives $P \not\in Y$ ... (vii)

Again from the exact sequence

$$(0) \rightarrow E' \rightarrow T \rightarrow T/E' \rightarrow (0)$$

(where $E' \subset T$) we have, from 2.3.12, $\mathcal{A}(T) \subseteq \mathcal{A}(E') \cup \mathcal{A}(T/E')$

So $\mathcal{A}(T) \subseteq X \cup \{P\}$, [by (i) and (v)] ... (viii)

Since $T \subseteq E$, by the definition, we get $\mathcal{A}(T) \subseteq \mathcal{A}(E)$.

Thus $\mathcal{A}(T) \subseteq X \cup Y$ ... (ix)

Therefore, $\mathcal{A}(T) \subseteq (X \cup \{P\}) \cap (X \cup Y)$, [by (viii) and (ix)]

$\Rightarrow \mathcal{A}(T) \subseteq (X \cap (X \cup Y)) \cup (\{P\} \cap (X \cup Y))$

$\Rightarrow \mathcal{A}(T) \subseteq X \cup (\{P\} \cap (X \cup Y))$

Now, if $P \not\in X$ then $\mathcal{A}(T) \subseteq X \cup \emptyset$, [by (vii)]

$\Rightarrow \mathcal{A}(T) \subseteq X.$

Again, if $P \in X$ then $\mathcal{A}(T) \subseteq X \cup \{P\} = X$

$\Rightarrow \mathcal{A}(T) \subseteq X$

Therefore $T \in \mathcal{H}$. It contradicts the maximal character of E' in $\mathcal{H}$ (since $E' \subset T$).

Thus we get, $Y = \mathcal{A}(E/E')$.

Also, (ii) gives, $\mathcal{A}(E) \subseteq \mathcal{A}(E') \cup Y$

$\Rightarrow X \cup Y \subseteq \mathcal{A}(E') \cup Y$

$\Rightarrow X \subseteq \mathcal{A}(E')$, as $X \cap Y = \emptyset$.

Therefore (i) gives, $\mathcal{A}(E') = X$. //

2.5.7. <u>Theorem</u> [[15]] : Let E be an fsd-1 N-group with acc on annihilators then $\mathcal{A}(E)$ is finite.

<u>Proof</u> : Let $P_1 \in \mathcal{A}(E)$ and $\mathcal{A}(E) = \{P_1\} \cup Y$ such that $P_1 \notin Y$. Then by 2.5.6. we get a supplement E' in E such that

$$\mathcal{A}(E') = \{P_1\} \qquad \ldots \qquad (i)$$

and $\mathcal{A}(E/E') = Y$

Thus, $\mathcal{A}(E) = \mathcal{A}(E') \cup \mathcal{A}(E/E')$.

Now, if $P_2 \in \mathcal{A}(E)$ and $P_2 \neq P_1$ then $P_2 \in \mathcal{A}(E') \cup \mathcal{A}(E/E')$. So $P_2 \in \mathcal{A}(E/E')$. Thus let us choose $P_2 = \text{Ann}(M/E')$ where M/E' is a prime N-subgroup of E/E'.

Again, E is a prime complete with non small extension and so M/E' has a non-small prime ideal extension H/E' in E/E' such that $M/E' \subseteq H/E'$. Thus $P_2 = \text{Ann}(M/E') = \text{Ann}(H/E')$.

Also, H/E' is a prime N-subgroup and so H/E' is primary. Hence $\mathcal{A}(H/E') = P_2$.

Here E being an fsd-1 N-group and hence E/E' is also with fsd-1 [by 2.4.2]. Since H/E' is a non-small ideal of E/E' then by 2.4.13. H/E' contains a non-zero supplement E''/E' in E/E'. Thus E''/E' is also prime as H/E' is prime.

Therefore, $\mathcal{A}(E''/E') = P_2 = \text{Ann}(E''/E') = \text{Ann}(H/E')$ ... (ii)

Again from the exact sequence

$$(0) \to E'' \to E \to E/E'' \to (0)$$

as above we get

$$\mathcal{A}(E) \subseteq \mathcal{A}(E'') \cup \mathcal{A}(E/E'') \quad \ldots \quad \text{(iii)}$$

Since E''/E' is non-zero, we get $E' \subset E''$. Thus the exact sequence

$$(0) \to E' \to E'' \to E''/E' \to (0)$$

gives $\mathcal{A}(E'') \subseteq \mathcal{A}(E') \cup \mathcal{A}(E''/E')$, [by 2.3.12]

$$\Rightarrow \mathcal{A}(E'') \subseteq \{P_1, P_2\}, \quad \text{[by (i) and (ii)]}$$

Hence (iii) gives

$\mathcal{A}(E) \subseteq \{P_1, P_2\} \cup \mathcal{A}(E/E'')$.

In this way, if we consider another distinct element $P_3 \in \mathcal{A}(E)$ then we get a supplement E''' in E such that $\mathcal{A}(E) \subseteq \{P_1, P_2, P_3\} \cup \mathcal{A}(E/E''')$ and $E' \subset E'' \subset E'''$.

Thus, if $\mathcal{A}(E)$ is an infinite set we get a strictly ascending infinite chain of supplements $E' \subset E'' \subset E''' \subset \ldots$ and this contradicts the proposition 2.4.21 . Hence $\mathcal{A}(E)$ must be finite. //

2.5.8. <u>Definition</u> [[15]] : <u>An s-p decomposition</u> of (0) of an fsd-1 N-group E is an expression $E_1 \cap E_2 \cap \ldots \cap E_t = (0)$ where each E_i $(i = 1,2,\ldots, t)$ is a supplement in E such that

(i) $E_1 \cap E_2 \cap \ldots \cap \hat{E}_i \cap \ldots \cap E_t \neq (0)$ for any i(symbol $\wedge$ indicates the exclusion of the set underneath it) and

(ii) each factor N-group E/E_i is primary with $\mathcal{A}(E/E_i) \neq \mathcal{A}(E/E_j)$ for $i \neq j$, where $1 \leq i, j \leq t$.

2.5.9. <u>Theorem</u> [[15]] : Let E be an fsd-1 N-group with acc on annihilators then the following results hold good :

(I) There exists an s-p decomposition of (0) in E.

(II) If $E_1 \cap E_2 \cap \ldots \cap E_t = (0)$ is an s-p decomposition of (0) in E then $\mathcal{A}(E) = \mathcal{A}(E/E_1) \cup \mathcal{A}(E/E_2) \cup \ldots \cup \mathcal{A}(E/E_t)$.

Proof : (I) We have by 2.5.7, $\mathcal{A}(E)$ is a finite set and let $\mathcal{A}(E) = \{P_1, P_2, \ldots, P_t\}$ ... (i)

Therefore, $\mathcal{A}(E) = \{P_1, P_2, \ldots, P_t\} \cup \{P_i\}$ $(1 \leq i \leq t)$

Thus by the theorem 2.5.6 we get a supplement E_i in E such that $\mathcal{A}(E_i) = \{P_1, P_2, \ldots, \hat{P_i}, \ldots, P_t\}$ and $\mathcal{A}(E/E_i) = \{P_i\}$. This is true for all values of i and each E/E_i is primary as $\mathcal{A}(E/E_i) = \{P_i\}$.

Again, as P_i's are distinct, $\mathcal{A}(E/E_i) \neq \mathcal{A}(E/E_j)$, for $i \neq j$, $1 \leq i, j \leq t$.

Now, let $Ann(H) \in \mathcal{A}(E_1 \cap E_2 \cap \ldots \cap E_t)$ for some prime N-subgroup H of $E_1 \cap E_2 \cap \ldots \cap E_t$. Then H is a prime N-subgroup of each of E_i $(i = 1, 2, \ldots, t)$.

So, $Ann(H) \in \mathcal{A}(E_i)$ for all i.

$\Rightarrow$ $Ann(H) \in \mathcal{A}(E_1) \cap \mathcal{A}(E_2) \cap \ldots \cap \mathcal{A}(E_t)$

$\Rightarrow$ $\mathcal{A}(E_1 \cap E_2 \cap \ldots \cap E_t) \subseteq \mathcal{A}(E_1) \cap \mathcal{A}(E_2) \cap \ldots \cap \mathcal{A}(E_t) = \emptyset$

$\Rightarrow$ $\mathcal{A}(E_1 \cap E_2 \cap \ldots \cap E_t) = \emptyset$

$\Rightarrow$ $E_1 \cap E_2 \cap \ldots \cap E_t = (0)$, [by 2.3.10]

If possible, let $E_1 \cap E_2 \cap \ldots \cap \hat{E_i} \cap \ldots \cap E_t = (0)$ for some i $(1 \leq i \leq t)$. We consider the mapping

$\alpha : E \to E/E_1 \oplus \cdots \oplus \widehat{E/E_i} \oplus \cdots \oplus E/E_t$ such that

$\alpha(e) = (e+E_1, \ldots, \widehat{e+E_i}, \ldots, e+E_t)$ for $e \in E$.

Clearly α is an N-homomorphism.

Now, if $\alpha(e) = \alpha(e')$ for $e, e' \in E$ then

$(e+E_1, \ldots, \widehat{e+E_i}, \ldots, e+E_t) = (e'+E_1, \ldots, \widehat{e'+E_i}, \ldots, e'+E_t)$

$\Rightarrow$ $e + E_j = e' + E_j$, for $j \neq i$

$\Rightarrow$ $e - e' \in E_j$, for all $j \neq i$.

$\Rightarrow$ $e - e' \in E_1 \cap E_2 \cap \cdots \cap \widehat{E_i} \cap \cdots \cap E_t = (0)$, by supposition

$\Rightarrow$ $e = e'$

Hence α is an N-monomorphism and thus E is embedded in $\bigoplus_{j \neq i}^{t} E/E_j$ when $\bigcap_{j\neq i}^{t} E_j = (0)$ for each i ($1 \leq i \leq t$). So E is considered as an N-subgroup of $\bigoplus_{j\neq i}^{t} E/E_j$ when $\bigcap_{j\neq i} E_j = (0)$. Thus a prime N-subgroup of E is also a prime N-subgroup of $\bigoplus_{j\neq i}^{t} E/E_j$ for each i ($1 \leq i \leq t$).

Therefore, $\mathcal{A}(E) \subseteq \mathcal{A}\left(\bigoplus_{j\neq i}^{t} E/E_j\right)$

$= \bigcup_{j\neq i}^{t} \mathcal{A}(E/E_j)$ for each i, [by 2.3.14]

$$\Rightarrow \mathcal{A}(E) \subseteq \mathcal{A}(E/E_1) \cup \mathcal{A}(E/E_2) \cup \ldots \cup \widehat{\mathcal{A}(E/E_i)} \cup \ldots \cup \mathcal{A}(E/E_t)$$

$$= \{P_1\} \cup \{P_2\} \cup \ldots \cup \widehat{\{P_i\}} \cup \ldots \cup \{P_t\}$$

$\Rightarrow \mathcal{A}(E) \subseteq \{P_1, P_2, \ldots, \hat{P}_i, \ldots, P_t\}$ which contradicts the result (i). Therefore, $E_1 \cap E_2 \cap \ldots \cap \hat{E}_i \cap \ldots \cap E_t \neq (0)$. //

(II) Let $\bigcap_{j=1}^{t} E_j = (0)$ be an s-p decomposition of (0) in E. Thus each E_j is a supplement in E. Here $E_1 \cap E_2 \cap \ldots \cap E_j \cap \ldots \cap E_t \neq (0)$ for each j and each factor N-group E/E_j is primary with $\mathcal{A}(E/E_i) \neq \mathcal{A}(E/E_j)$ for each $j \neq i$.

Let us consider the mapping $\alpha : E \to \bigoplus_{j=1}^{t} E/E_j$ such that $\alpha(e) = (e+E_1, \ldots, e+E_t)$ for $e \in E$.

It can be easily verified that α is an N-monomorphism. Thus E is embedded in $\bigoplus_{j=1}^{t} E/E_j$. So, E is regarded as an N-subgroup of $\bigoplus_{j=1}^{t} E/E_j$. Thus a prime N-subgroup of E is also a prime N-subgroup of $\bigoplus_{j=1}^{t} E/E_j$. Hence by 2.3.14, we get

$$\mathcal{A}(E) \subseteq \mathcal{A}\left(\bigoplus_{j=1}^{t} E/E_j\right) = \bigcup_{j=1}^{t} \mathcal{A}(E/E_j) \qquad \ldots \quad (i)$$

$$\Rightarrow \mathcal{A}(E) \subseteq \mathcal{A}(E/E_1) \cup \mathcal{A}(E/E_2) \cup \dots \cup \widehat{\mathcal{A}(E/E_i)} \cup \dots \cup \mathcal{A}(E/E_t)$$

$$= \{P_1\} \cup \{P_2\} \cup \dots \cup \widehat{\{P_i\}} \cup \dots \cup \{P_t\}$$

$\Rightarrow \mathcal{A}(E) \subseteq \{P_1, P_2, \dots, \hat{P}_i, \dots, P_t\}$ which contradicts the result (i). Therefore, $E_1 \cap E_2 \cap \dots \cap \hat{E}_i \cap \dots \cap E_t \neq (0)$. //

(II) Let $\bigcap_{j=1}^{t} E_j = (0)$ be an s-p decomposition of (0) in E. Thus each E_j is a supplement in E. Here $E_1 \cap E_2 \cap \dots \cap E_j \cap \dots \cap E_t \neq (0)$ for each j and each factor N-group E/E_j is primary with $\mathcal{A}(E/E_i) \neq \mathcal{A}(E/E_j)$ for each $j \neq i$.

Let us consider the mapping $\alpha : E \to \bigoplus_{j=1}^{t} E/E_j$ such that $\alpha(e) = (e+E_1, \dots, e+E_t)$ for $e \in E$.

It can be easily verified that α is an N-monomorphism. Thus E is embedded in $\bigoplus_{j=1}^{t} E/E_j$. So, E is regarded as an N-subgroup of $\bigoplus_{j=1}^{t} E/E_j$. Thus a prime N-subgroup of E is also a prime N-subgroup of $\bigoplus_{j=1}^{t} E/E_j$. Hence by 2.3.14, we get

$$\mathcal{A}(E) \subseteq \mathcal{A}\left(\bigoplus_{j=1}^{t} E/E_j\right) = \bigcup_{j=1}^{t} \mathcal{A}(E/E_j) \qquad \dots \quad (i)$$

Now, consider a mapping $\beta : \bigcap_{j\neq i}^{t} E_j \rightarrow E/_{E_i}$ for each i ($1 \leq i \leq t$) such that $\beta(e) = e + E_i$, $e \in E_j$ for all $j \neq i$.

Clearly, β is an N-homomorphism.

If $\beta(e) = \beta(e')$ for $e, e' \in \bigcap_{j\neq i}^{t} E_j$. Then

$$e + E_i = e' + E_i$$

$$\Rightarrow \quad e - e' \in E_i$$

But $e - e' \in \bigcap_{j\neq i}^{t} E_j$. Thus $e - e' \in \bigcap_{j=1}^{t} E_j = (0)$.

Therefore, $e = e'$.

Hence β is an N-monomorphism. Thus $\bigcap_{j\neq i}^{t} E_j$ is embedded in $E/_{E_i}$. So $\bigcap_{j\neq i}^{t} E_j$ is an N-subgroup of $E/_{E_i}$. Thus a prime N-subgroup of $\bigcap_{j\neq i}^{t} E_j$ is also a prime N-subgroup of $E/_{E_i}$.

$$\text{Hence } \mathcal{A}\left(\bigcap_{j\neq i}^{t} E_j\right) \subseteq \mathcal{A}\left(E/_{E_i}\right) \quad \cdots \quad \text{(ii)}$$

Now, $\bigcap_{j\neq i}^{t} E_j \neq (0) \Rightarrow \mathcal{A}\left(\bigcap_{j\neq i}^{t} E_j\right) \neq \emptyset$.

Since $\mathcal{A}(E/_{E_j})$ is singleton, (ii) gives $\mathcal{A}\left(\bigcap_{j\neq i}^{t} E_j\right) = \mathcal{A}(E/_{E_i})$

$$\text{Therefore, } \bigcup_{i=1}^{t} \mathcal{A}\left(E/_{E_i}\right) = \bigcup_{i=1}^{t} \mathcal{A}\left(\bigcap_{j\neq i}^{t} E_j\right) \quad \cdots \quad \text{(iii)}$$

Now, consider a mapping $\beta : \bigcap_{j\neq i}^{t} E_j \rightarrow E/E_i$ for each i ($1 \leq i \leq t$) such that $\beta(e) = e + E_i$, $e \in E_j$ for all $j \neq i$.

Clearly, β is an N-homomorphism.

If $\beta(e) = \beta(e')$ for $e, e' \in \bigcap_{j\neq i}^{t} E_j$. Then

$$e + E_i = e' + E_i$$

$$\Rightarrow \quad e - e' \in E_i$$

But $e - e' \in \bigcap_{j\neq i}^{t} E_j$. Thus $e - e' \in \bigcap_{j=1}^{t} E_j = (0)$.

Therefore, $e = e'$.

Hence β is an N-monomorphism. Thus $\bigcap_{j\neq i}^{t} E_j$ is embedded in E/E_i . So $\bigcap_{j\neq i}^{t} E_j$ is an N-subgroup of E/E_i. Thus a prime N-subgroup of $\bigcap_{j\neq i}^{t} E_j$ is also a prime N-subgroup of E/E_i .

Hence $\mathcal{A}(\bigcap_{j\neq i}^{t} E_j) \subseteq \mathcal{A}(E/E_i)$ ... (ii)

Now, $\bigcap_{j\neq i}^{t} E_j \neq (0) \Rightarrow \mathcal{A}(\bigcap_{j\neq i}^{t} E_j) \neq \emptyset$.

Since $\mathcal{A}(E/E_j)$ is singleton, (ii) gives $\mathcal{A}(\bigcap_{j\neq i}^{t} E_j) = \mathcal{A}(E/E_i)$

Therefore, $\bigcup_{i=1}^{t} \mathcal{A}(E/E_i) = \bigcup_{i=1}^{t} \mathcal{A}(\bigcap_{j\neq i}^{t} E_j)$ ... (iii)

Since E_i's are supplements in E, $\bigcap_{j\neq i}^{t} E_j \subseteq E$ and $\mathcal{A}(\bigcap_{j\neq i}^{t} E_j)$ $\subseteq \mathcal{A}(E)$ for each i.

Thus, $\bigcup_{i=1}^{t} \mathcal{A}(\bigcap_{j\neq i}^{t} E_j) \subseteq \mathcal{A}(E)$

$$\Rightarrow \bigcup_{i=1}^{t} \mathcal{A}(E/E_i) \subseteq \mathcal{A}(E), \quad \text{[by (iii)]} \quad \ldots \quad \text{(iv)}$$

Hence (i) and (iv) give

$$\mathcal{A}(E) = \bigcup_{j=1}^{t} \mathcal{A}(E/E_j) \; . \; /\!/$$

2.5.10. <u>Theorem</u> [[15]] : Let E be an fsd-1 N-group with acc on annihilators such that $Z_1(E) = (0)$. If a commutative ring N (with unity) has no infinite direct sum of left ideals and every essential left ideal of N is essential N-subgroup of ${}_N N$ then for any $x \in \bigcap_{P \in \mathcal{A}(E)} P$, there exists a $t \in Z^+$ such that $x^t \in \text{Ann}(E)$.

<u>Proof</u> : Let $x \in \bigcap_{P \in \mathcal{A}(E)} P$ then $x \in P = \text{Ann}(M)$ (say), where M is a prime N-subgroup of E.

Let $\phi_i : E \to E$ be a mapping such that

$$\phi_i(e) = x^i e, \text{ for } e \in E \; (i = 1,2,3,\ldots)$$

N being a commutative ring, so each $x \in N$ is distributive

$\Rightarrow \quad r_E(\mathrm{Ann}(r_E(x^t))) = r_E(\mathrm{Ann}(r_E(x^{t+1})))$

$\Rightarrow \quad r_E(x^t) = r_E(x^{t+1})$, [by 1.3.19] ... (iii)

Now, $e \in \mathrm{Ker}\, \phi_{t+1} \Rightarrow \phi_{t+1}(e) = 0$

$\Rightarrow \quad x^{t+1}e = 0$

$\Rightarrow \quad e \in r_E(x^{t+1}) = r_E(x^t)$, [by (iii)]

$\Rightarrow \quad x^t e = 0$

$\Rightarrow \quad e \in \mathrm{Ker}\, \phi_t$.

$\Rightarrow \quad \mathrm{Ker}\, \phi_{t+1} \subseteq \mathrm{Ker}\, \phi_t$. Also, (ii) gives $\mathrm{Ker}\, \phi_t \subseteq \mathrm{Ker}\, \phi_{t+1}$

Hence $\mathrm{Ker}\, \phi_t = \mathrm{Ker}\, \phi_{t+1}$ ... (iv)

We note that x^tE is an N-subgroup of E. For, if $e_1, e_2 \in E$ then $x^te_1, x^te_2 \in x^tE$ which gives $x^te_1 - x^te_2 \in x^tE$ [by (i)]. Also $n\,(x^tE) = (nx^t)\,E = (x^tn)\,E = x^t(nE) \subseteq x^tE$ as N is commutative.

Consider a map $f : x^tE \to x^tE$ such that $f(x^te) = x^{t+1}e$, for $e \in E$.

Thus, for $e, e_1 \in E$, we have

$$
\begin{aligned}
f(x^te + x^te_1) &= f(x^t(e+e_1)), && \text{[by (i)]} \\
&= x^{t+1}(e + e_1), && \text{(by definition)} \\
&= x^{t+1}e + x^{t+1}e_1, && \text{[by (i)]} \\
&= f(x^te) + f(x^te_1)
\end{aligned}
$$

Also, for $n \in N$, $f(n(x^t e))$

$= f((n\, x^t)e)$

$= f((x^t n)e)$, as N is commutative

$= f(x^t(ne))$

$= x^{t+1}(ne)$, by definition

$= n\,(x^{t+1}e)$ as N is commutative

$= n\, f(x^t e)$.

Hence f is an N-homomorphism.

Again, if $f(x^t e) = f(x^t e_1)$ then

$$x^{t+1}e = x^{t+1}e_1$$

$\Rightarrow \quad x^{t+1}(e - e_1) = 0, \quad$ [by (i)]

$\Rightarrow \quad e - e_1 \in \text{Ker}\,\phi_{t+1} = \text{Ker}\,\phi_t$

$\Rightarrow \quad x^t(e - e_1) = 0$

$\Rightarrow \quad x^t e = x^t e_1, \quad$ [by (i)]

Therefore f is injective.

Also, we have, $x^t E \subseteq E$ and so

$$\mathcal{A}(x^t E) \subseteq \mathcal{A}(E).$$

If $x^t E = (0)$ then $x^t \in \text{Ann}\,(E)$, we are done.

Suppose $x^t E \neq (0)$ then $\mathcal{A}(x^t E) \neq \phi$. Then there exists a non-zero prime N-subgroup E' of $x^t E$ such that $\mathrm{Ann}(E') \in \mathcal{A}(x^t E)$. Since $x \in P$, for each $P \in \mathcal{A}(E)$, we get $x \in P$ for each $P \in \mathcal{A}(x^t E)$. So $x \in \mathrm{Ann}(E')$ which gives $xE' = (0)$. That is $f(E') = (0)$. Again, f is injective, it follows that $E' = (0)$, a contradiction. Hence $x^t E = (0)$ and so $x^t \in \mathrm{Ann}(E)$. //

CHAPTER III

Near-rings with acc on left annihilators

We have proved some results on (right) near-rings satisfying the ascending chain condition (acc) on left annihilators. In particular, we confine our discussion on left Goldie near-rings which is defined as a (right) near-ring with acc on left annihilators and having no infinite direct sum of left ideals. Here we give another proof of theorem (7) of A.Oswald [41] using non - singularity of such a near-ring. This result, published in National Academy Science letters [[13]], plays a key role in very many results of this chapter. We have shown the existence (or coincidence) of (right) near-ring of left quotients of a strictly left Goldie near-ring with some interesting characters ([[37]], IJPAM).

The first one of the five sections of this chapter contains preliminary results required for what follows. The second contains some basic results on strongly semiprime near-rings. An important result of the section reads as : If N is a strongly semiprime strictly left Goldie near-ring such that an essential left ideal of N is an essential left N-subgroup of it then N satisfies the dcc on left annihilators. Properties of maximal annihilators of such a near-ring N with acc on left annihilators find their places in the third section.

In the fourth section, we show that the collection Γ of all maximal left annihilators in a strongly semiprime strictly left Goldie near-ring is a finite one in some special cases and is such that $\bigcap_{P \in \Gamma} P = (0)$. It is also seen that such a near-ring N is embedded in $\bigoplus_{P_i \in \Gamma} N/P_i$, a direct sum of strongly prime strictly left Goldie near-rings.

The last section contains results on what are known as classical and complete near-ring of left quotients of a strongly semiprime strictly left Goldie near-ring N. In such a d.g.nr. N with distributively generated left annihilators, the complete near-ring Q(N) of left quotients of N is a classical near-ring Q of left quotients with respect to (w.r.t.) a set S of distributive non-zero-divisors. Moreover, Q has no nilpotent left Q-subgroups and it satisfies the dcc on its left Q-subgroups. Conversely, if Q is a strictly left Goldie near-ring so also N and it has no non-zero nilpotent left N-subgroups when Q satisfies the dcc on its left Q-subgroups.

3.1. Prerequisites

3.1.1. Lemma : Let x be any element of a near-ring N such that $l(x^{t+1}) \cap Nx^t \neq (0)$ for all $t \in Z^+$ then

$$l(x) \subset l(x^2) \subset l(x^3) \subset \ldots \subset l(x^t) \subset \ldots.$$

is a strictly ascending chain.

<u>Proof</u> : Suppose there exists a $t \in Z^+$ such that

$$l(x^t) = l(x^{t+1}) \qquad \ldots \qquad (i)$$

Then by 1.3.22, we have $l(x^{t+m}) = l(x^t)$ for all $m \in Z^+$.

Let for $n \in N$, $\alpha = nx^t \ (\neq 0) \in l(x^{t+1}) \cap Nx^t$.

Then $\alpha \in l(x^t)$ [by (i)]

$\Rightarrow \alpha x^t = 0$

$\Rightarrow nx^{2t} = 0$

$\Rightarrow n \in l(x^{2t}) = l(x^t)$ [by 1.3.22]

$\Rightarrow \alpha = nx^t = 0$, a contradiction.

Therefore, $l(x^t) \subset l\,(x^{t+1})$, for all $t \in Z^+$. //

<u>3.1.2. Note</u> : In what follows, because of 1.2.14 we shall write " left N-subgroup A of N" instead of writing N-subgroup A of ${}_N N$.

<u>3.1.3. Definitions</u> : A non-zero left N-subgroup A of a near-ring N is called an <u>essential left N-subgroup</u> of a left N-subgroup B of N (denoted $A \subseteq_e B$) if for every non-zero left N-subgroup $C (\subseteq B)$ of N, $A \cap C \neq (0)$.

If $A \subseteq_e B$ then we say, B is an <u>essential extension</u> of A.

For the sake of completeness, we restate the lemmas 2.2.3 and 2.2.4. as follows.

3.1.4. Lemma : If A,B,C are left N-subgroups of N with $A \subseteq_e B \subseteq_e C$ then $A \subseteq_e C$.

3.1.5. Lemma : If A,B,C are left N-subgroups of N with $A \subseteq B \subseteq C$, $A \subseteq_e C$ then $A \subseteq_e B \subseteq_e C$.

3.1.6. Lemma [13] : If A and B are two left N-subgroups of N with $B \subseteq_e A$, then for any $a\ (\neq 0) \in A$, there exists a left N-subgroup $L \subseteq_e N$ such that $La \subseteq B$, $La \neq (0)$.

3.1.7. Lemma [37] : Let I and J be two left N-subgroups of N. If $I \subseteq_e J$ then $(I ; x)\ [= \{n \in N \mid nx \in I\}] \subseteq_e N$ for each $x \in J$.

Proof : By 1.2.22, $(I ; x)$ is a left N-subgroup of N. Let H be a non-zero left N-subgroup of N.

$$\text{Now, } Hx = (0) \Rightarrow hx = 0 \in I, \text{ for each } h\ (\neq 0) \in H.$$

$$\Rightarrow h \in H \cap (I ; x)$$

$$\Rightarrow H \cap (I ; x) \neq (0) .$$

Now, since $Hx \subseteq J$, $I \subseteq_e J$ and

if $Hx \neq (0)$ then $Hx \cap I \neq (0)$.

Let $h_1x\ (\neq 0) \in I$ for $h_1 \in H$.

Then $h_1 \in (I ; x)$

$\Rightarrow h_1 \in H \cap (I ; x)$

If $h_1 = 0$ then $h_1 x = 0$, a contradiction.

So, $h_1 \neq 0$ giving thereby $H \cap (I ; x) \neq (0)$.

Therefore $(I ; x) \subseteq_e N$. //

If $J = N$ then as a corollary to the above, we get

3.1.8. Corollary [[14]]: Let I be a left N-subgroup of N with $I \subseteq_e N$ then for any $x \in N$, $(I ; x) \subseteq_e N$. //

3.1.9.(a) Lemma : Let E_1 be an ideal of an N-group E and $f : E \to E/_{E_1}$ is the natural N-epimorphism. If X is an essential N-subgroup of $E/_{E_1}$ then $f^{-1}(X)$ is an essential N-subgroup of E.

Proof : Let M be an N-subgroup of E such that

$$f^{-1}(X) \cap M = (0) .$$

Then, $\text{Ker } f \cap M \subseteq f^{-1}(X) \cap M = (0)$.

$\Rightarrow$ $\text{Ker } f \cap M = (0)$

$\Rightarrow$ $f|_M$ is injective

$\Rightarrow$ $f(M) \cong M$.

Now, if $x \in X \cap f(M)$ then $x = f(m) \in X$ for some $m \in M$.

So, $m \in f^{-1}(X) \cap M = (0)$

$\Rightarrow$ $x = f(m) = 0$

$\Rightarrow$ $X \cap f(M) = (0)$

$\Rightarrow$ $f(M) = (0)$, (for $X \subseteq_e E/E_1$)

$\Rightarrow$ $M \cong f(M) = (0)$

$\Rightarrow$ $f^{-1}(X) \subseteq_e E$. //

If P is an ideal of N then taking E = N, the following result follows as a corollary to the above.

3.1.9(b) Corollary : Let P be an ideal of N and $f : N \to N/P$ is the natural epimorphism. If A is an essential left $\overline{N}$-subgroup of $\overline{N}$ (= N/P) then $f^{-1}(A)$ is an essential left N-subgroup of N.

By note 3.1.?., it therefore follows that

3.1.9(c) Note : If P is an ideal of N and A/P is an essential left $\overline{N}$ (= N/P) subgroup of $\overline{N}$, then A is an essential left N-subgroup of N.

3.1.10. Lemma : $Z_1(N)$ (= $\{x \in N \mid Ax = (0)$, for some essential left N-subgroup A of N $\}$) is an invariant subset of N.

Proof : Let $x \in Z_1(N)$. Then $Ax = (0)$, for some essential left N-subgroup A of N. So, by 3.1.6, for any $n\ (\neq 0) \in N$ there exists an essential left N-subgroup L of N such that $Ln \subseteq A$, $Ln \neq (0)$.

This gives, $L(nx) = (Ln)x \subseteq Ax = (0)$.

$\Rightarrow$ $nx \in Z_1(N)$.

And, $A(xn) = (Ax)\,n = (0)$

$\Rightarrow xn \in Z_1(N)$.

Hence $Z_1(N)$ is an invariant subset of N. //

3.1.11. Definitions : A near-ring N is called left non-singular if $Z_1(N) = (0)$. A near-ring N is called left singular if $Z_1(N) = N$.

3.1.12. Lemma : An element $x \in Z_1(N)$ if and only if $l(x) \subseteq_e N$.

Proof : Similar to that of 2.2.5.

3.1.13. Lemma [[13]] : If I is a left N-subgroup of a left non-singular near-ring N such that $l(B) \subseteq_e I$ then $l(B) = I$.

Proof : Similar to that of 2.2.6.

3.1.14. Lemma : If N satisfies the acc on left annihilators then $Z_1(N)$ is a nil invariant subset of N.

Proof : Let $x \in Z_1(N)$, then by 3.1.12, $l(x) \subseteq_e N$.

Now, $l(x) \subseteq l(x^t) \subseteq l(x^{t+1})$, for all $t \in Z^+$.

As $l(x) \subseteq_e N$, we have by 3.1.5, $l(x^{t+1}) \subseteq_e N$.

So, $l(x^{t+1}) \cap Nx^t \neq (0)$, if $x^t \neq 0$ (since $1 \in N$ and Nx^t is a non-zero left N-subgroup of N).

$$\Rightarrow \quad l(x) \subset l(x^2) \subset \ldots \subset l(x^t) \subset l(x^{t+1}) \subset \ldots \quad \text{(by 3.1.1.)}$$

and thus we meet a contradiction.

Therefore, $x^t = 0$ for some $t \in Z^+$, in otherwords x is nilpotent.

Hence by 3.1.10, $Z_1(N)$ is a nil invariant subset of N. //

3.1.15. Definition : A left N-subgroup A of N is said to be a weakly essential left N-subgroup of N if for any left ideal $I(\neq 0)$ of N, $A \cap I \neq (0)$.

It is to be noted that an essential left N-subgroup of N is a weakly essential left N-subgroup of it. However, the following example shows that the converse of it is not true.

3.1.16. Example (H(37), Page 341-342 [42]) : Consider the near-ring $S_3 = \{0,a,b,c,x,y\}$ with operation addition [defined in table 1.3(i)] and multiplication defined by the following table .

·	0	a	b	c	x	y
0	0	0	0	0	0	0
a	0	a	b	c	0	0
b	0	a	b	c	0	0
c	0	a	b	c	0	0
x	0	0	0	0	0	0
y	0	0	0	0	0	0

Table : 3.1

Here non-zero left S_3-subgroups are $\{0,a\}$, $\{0,b\}$, $\{0,c\}$, $\{0,x,y\}$ and S_3. $\{0,x,y\}$ and S_3 are the only non-zero left ideals. This shows that the S_3-subgroup $\{0,x,y\}$ is weakly essential but not an essential left S_3-subgroup.

3.2. Strongly semiprime near-rings

Here, in the beginning, it is worth mentioning that the idea of a strictly prime ideal as defined in the preceding chapter is basically different from what we want to introduce here some other types of algebraic substructures in particular a strongly prime ideal, strongly semi-prime near-ring etc.

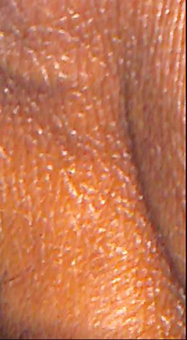
3.2.1. Definitions : An ideal I of N is called a strongly prime ideal if, for two non-zero invariant subsets A and B, $AB \subseteq I$ implies either $A \subseteq I$ or $B \subseteq I$.

A near-ring N is strongly prime if (0) is a strongly prime ideal of N.

3.2.2. Definitions : An ideal I of N is a prime ideal of N if for any two non-zero ideals A and B, $AB \subseteq I$ implies either $A \subseteq I$ or $B \subseteq I$. A near- ring N is prime if (0) is a prime ideal of N.

It is to be noted that every strongly prime ideal is prime and every strongly prime near-ring is also prime.

3.2.3. Definitions : A near-ring N is called a strongly



<u>semiprime</u> near-ring if N has no non-zero nilpotent invariant subset of N.

Hence a strongly semiprime near-ring has no non-zero nilpotent invariant subnear-ring also.

Near-ring N is said to be <u>semiprime</u> if N has no non-zero nilpotent ideals of N.

We note that a strongly semiprime near-ring is also semiprime. A partial converse of it is seen in A. Oswald [41]. But, in general, the converse is not true. It becomes clear from the following.

3.2.4. Example (J(84), Page 342-343 [42]) :

N = {0,1,2,3,4,5,6,7} is a near-ring under addition modulo 8 and multiplication defined by the following table.

.	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	0	1	1	1	1	1
2	0	2	0	2	2	2	2	2
3	0	3	0	3	3	3	3	3
4	0	4	0	4	4	4	4	4
5	0	5	0	5	5	5	5	5
6	0	6	0	6	6	6	6	6
7	0	7	0	7	7	7	7	7

Table : 3.2

Here N has no non-zero ideal except N itself. Table shows that N is not nilpotent. Thus N is a semiprime near-ring. On the otherhand, N has a non-zero invariant subset $\{0,2\}$ which is nilpotent. Thus N is not a strongly semiprime near-ring.

Therefore, a semiprime near-ring need not be strongly semiprime. //

3.2.5. Lemma [[14]] : A strongly semiprime near-ring N has no non-zero nilpotent left (right) N-subset of N.

Proof : If A is a non-zero nilpotent left N-subset of N then, for some $t \in Z^+$, $A^t = (0)$ with $A^{t-1} \neq (0)$.

Write $B = A^{t-1}$ then

$$B^2 \subseteq A^{t-1} A = A^t = (0)$$

$$\Rightarrow B^2 = 0$$

Moreover, $(BN)\,N = B(N\,N) \subseteq BN$ and $N(BN) = (NB)N \subseteq BN$ (for B is a left N-subset of N).

Also, $(BN)^2 = (BN)(BN) = B(NB)N \subseteq BBN = B^2N = (0)$

N being strongly semiprime, it follows that $BN = (0)$ which gives $B = (0)$ (as $1 \in N$), a contradiction.

Therefore, N has no non-zero nilpotent left N-subset. In like manner, the result holds good for right N-subset of N also. //

3.2.6. Lemma : If I is an invariant subset of a strongly semiprime near-ring N then $l(I) = r(I)$.

Proof : We have, $l(I)\, I = (0)$

So, $(I\, l(I))^2 = (I\, l(I))\,(I\, l(I))$

$= I(l(I)I)\, l(I)$

$= (0)$,

$\Rightarrow$ $I\, l(I) = (0)$, (by 3.2.5 as $I\, l(I)$ is a left N-subset).

$\Rightarrow$ $l(I) \subseteq r(I)$

Similarly, $r(I)I = (0)$ gives $r(I) \subseteq l(I)$.

Hence $r(I) = l(I)$. //

3.2.7. Lemma [14] : A strongly semiprime near-ring N with acc on left annihilators has no non-zero nil left N-subset of N.

Proof : Let A be any non-zero left N-subset of N. Since N satisfies the acc on left annihilators, we can choose $a\ (\neq 0) \in A$ with $l(a)$ as large as possible.

Now, $a\, Na = (0)$

$\Rightarrow$ $(Na)^2 = (Na)(Na) = N(aNa) = (0)$

And Na being a non-zero left N-subset of $N(1 \in N,\ a \neq 0)$, we meet a contradiction to 3.2.5.

So, $aNa \neq (0)$.

Let $x \in N$ be such that $axa \neq 0$

Now, $xa \neq 0$ (otherwise $axa = 0$)

$\Rightarrow \quad x \notin l(a)$

Again, $z \in l(a) \Rightarrow za = 0$

$\Rightarrow z(axa) = (za)xa = 0$

$\Rightarrow z \in l(axa)$

$\Rightarrow l(a) \subseteq l(axa)$

But $l(a)$ being maximal, $l(axa) = l(a)$

So, $x \notin l(axa)$

$\Rightarrow x(axa) \neq 0$

$\Rightarrow (xa)^2 \neq 0$

$\Rightarrow (xax)a \neq 0$

$\Rightarrow xax \notin l(a) = l(axa)$

$\Rightarrow (xax)(axa) \neq 0$

$\Rightarrow (xa)^3 \neq 0$ and so on.

Thus, $(xa)^t \neq 0$, for any $t \in Z^+$.

Therefore, A possesses a non-zero non nilpotent element xa.

So A is not nil.

Hence N does not have any non-zero nil left N-subset of N. //

3.2.8. Lemma [[13,14]] : If N is a strongly semiprime near-ring with acc on left annihilators then N is left non-singular.

Proof : By 3.1.14, $Z_1(N)$ is a nil invariant subset of N and by 3.2.7, it follows that $\mathcal{Z}_1(N) = (0)$. Thus the result follows. //

Extending the idea of finite Goldie dimension (fgd) in near-ring given by Satyanarayana in [44], we define a near-ring to be of strictly finite Goldie dimension (strictly fgd) if N has no infinite independent family of left N-subgroups of N. Clearly a near-ring with strictly finite Goldie dimension is with finite Goldie dimension.

3.2.9. Lemma [[14,37]] : Let N be of strictly finite Goldie dimension. Then for $x \in N$, $Nx \subseteq_e N$ if $l(x) = (0)$.

Proof : Let A be a non-zero left N-subgroup of N such that $A \cap Nx = (0)$ when $l(x) = (0)$ for $x \in N$.

Now, each Ax^i ($i = 1,2,3,\ldots$) is a left N-subgroup of N.

Let $\alpha \in Ax^t \cap (A + Ax + \ldots + \widehat{Ax^t} + \ldots + Ax^s)$ where $0 \leq t \leq s$, $s \in Z^+$ and $\widehat{Ax^t}$ means exclusion of the term Ax^t of the sum.

Then $\alpha = a_t x^t = a + a_1 x + \ldots + \widehat{a_t x^t} + \ldots + a_s x^s$; for $a, a_1, a_2, \ldots, a_s \in A$.

$$\Rightarrow a = a_t x^t - a_s x^s - \ldots - a_2 x^2 - a_1 x$$

$$= (a_t x^{t-1} - a_s x^{s-1} - \ldots - a_2 x - a_1)x \in A \cap Nx = (0).$$

$\Rightarrow a_t x^{t-1} - a_s x^{s-1} - \ldots - a_1 = 0$ [as $l(x) = (0)$]

$\Rightarrow a_1 = (a_t x^{t-2} - a_s x^{s-2} - \ldots - a_2)x \in A \cap Nx = (0)$

$\Rightarrow a_t x^{t-2} - a_s x^{s-2} - \ldots - a_2 = 0$

Thus, continuation of the process will give us

$a = a_1 = a_2 = \ldots = a_s = 0$. So, $\alpha = 0$.

Hence $Ax^t \cap (A + Ax + \ldots + \widehat{Ax^t} + \ldots + Ax^s) = (0)$.

This is true for all $s,t \in Z^+$ with $0 \leq t \leq s$.

Therefore, $\{A, Ax, \ldots, Ax^s\}$ is an independent family for each $s \in Z^+$. In other words, $\{A, Ax, \ldots\}$ is an infinite indepdndent family of left N-subgroups and this contradicts the strictly finite Goldie dimension character of N.

Hence, $A \cap Nx \neq (0)$ proving thereby $Nx \subseteq_e N$. //

3.2.10. <u>Definition</u> : An element $x \in N$ is said to be a <u>non-zero-divisor</u> if $l(x) = (0) = r(x)$.

3.2.11. <u>Lemma</u> : Let N be left non-singular with acc on left annihilators and $x \in N$ is such that the left N-subgroup $Nx \subseteq_e N$. Then x is a non-zero-divisor.

<u>Proof</u> : Let $y \in N$ and A,B be two left N-subgroups of N such that $A \subseteq_e B$. We first prove that $Ay \subseteq_e By$.

For this, let C ($\neq$ 0) be a left N-subgroup of By. Then choose c(ε C) = b_1y ($\neq$ 0), for some b_1 ε B. Now as $A \subseteq_e B$, we get a left N-subgroup $L \subseteq_e N$, by 3.1.6, such that

$L\, b_1 \subseteq A$, $L\, b_1 \neq (0)$.

So, $L\, b_1y \subseteq Ay$. And as N is left non-singular, we get $L\,(b_1y) \neq (0)$ (otherwise $b_1y = 0$, not true).

Again, $L\, b_1y \subseteq N\, b_1y \subseteq NC \subseteq C$ gives $L\, b_1y \subseteq Ay \cap C$. As $L\, b_1y \neq (0)$, it therefore follows that $Ay \cap C \neq (0)$. Hence $Ay \subseteq_e By$.

We now write, A = Nx and B = N. Using the above fact and $Nx \subseteq_e N$, we get

$$Nx^2 \subseteq_e Nx \subseteq_e N.$$

By 3.1.4, we thus get, $Nx^2 \subseteq_e N$.

Similarly, we have $Nx^t \subseteq_e N$, for each $t \in Z^+$.

Now, N being with acc on left annihilators, the descending chain

$$l(x) \subseteq l(x^2) \subseteq l(x^3) \subseteq \ldots \text{ gives a } t \in Z^+$$

such that $l(x^t) = l(x^{t+1})$.

Now, if $l(x) \neq (0)$, then

$$Nx^t \subseteq_e N \Rightarrow Nx^t \cap l(x) \neq (0)$$

Choose z ($\neq$ 0) ε $Nx^t \cap l(x)$. Then $z = nx^t$, for $n \in N$ and $zx = 0$.

So, $nx^{t+1} = 0$

$\Rightarrow \quad n \in l(x^{t+1}) = l(x^t)$

$\Rightarrow \quad z = nx^t = 0$

$\Rightarrow \quad Nx^t \cap l(x) = (0)$, a contradiction.

Hence $l(x) = (0)$

Next, suppose $\alpha \in r(x)$.

Then, $x\alpha = 0$

$\Rightarrow \quad (Nx)\,\alpha = N(x\alpha) = (0)$

$\Rightarrow \quad \alpha \in Z_1(N) = (0)$ (as $Nx \subseteq_e N$ and N is left non-singular)

$\Rightarrow \quad \alpha = 0$

$\Rightarrow \quad r(x) = 0$

Hence x is a non-zero-divisor. //

3.2.12. <u>Definitions</u> : A near-ring N is called <u>left Goldie near-ring</u> if it has no infinite direct sum of left ideals (N is of finite Goldie dimension) and N satisfies the acc on left annihilators.

N is <u>strictly left Goldie</u> if N has no infinite independent family of left N-subgroups (N is of strictly fgd) and N satisfies the acc on left annihilators.

It is easy to see that a strictly left Goldie near-ring is left Goldie.

Since a left Goldie near-ring satisfies the acc on left annihilators, by 3.1.1, it follows that there is a $t \in Z^+$ for which $l(x^{t+1}) \cap Nx^t = (0)$.

Thus, we get

3.2.13 Remark : If N is left Goldie, then for any $x \in N$, we get a $t \in Z^+$ such that $l(x^{t+1}) \cap Nx^t = (0)$.

3.2.14. Proposition : If N is a strongly semiprime strictly left Goldie near-ring, then x ($\in N$) is a non-zero-divisor if and only if $Nx \subseteq_e N$.

Proof : As N is strongly semiprime and is with acc on left annihilators, by 3.2.8. N is left non-singular.

And therefore, by 3.2.9, as N is strictly left Goldie, x is a non-zero-divisor implies $Nx \subseteq_e N$.

Conversely assume $Nx \subseteq_e N$. Then by 3.2.11 we get x is a non-zero-divisor. //

We note that when we consider the near-ring N is an N-group N^N, then $Z_1(E)$ coincides with $Z_1(N)$. And if N is strongly semiprime then $Z_1(N) = (0)$ by 3.2.8. Again N being strictly left Goldie, it is left Goldie. So it has no infinite direct sum of left ideals. And therefore as a special case of 2.2.8, we get the following (as appeared in [[13]], Nat. Acad. Sci. Letters.)

3.2.15. Proposition [[13]] : Let N be a strongly semiprime strictly left Goldie near-ring such that essential left ideal is an essential left N-subgroup of N. Then N satisfies the dcc on left annihilators. //

It is to be noted that an essential left N-subgroup of N is also weakly essential. The converse is not true as shown in example 3.1.16. However, the following example is sufficient to show the existence of near-rings where every weakly essential left N-subgroup is also essential.

3.2.16. Example (J(91), Page 343 [42]):

N = {0,1,2,3,4,5,6,7} is a near-ring under addition modulo 8 and multiplication defined by the following table.

.	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	0	3	4	3	0	1
2	0	2	0	6	0	6	0	2
3	0	3	0	1	4	1	0	3
4	0	4	0	4	0	4	0	4
5	0	5	0	7	4	7	0	5
6	0	6	0	2	0	2	0	6
7	0	7	0	5	4	5	0	7

Table : 3.3 .

Here $\{0,4\}$ and $\{0,2,4,6\}$ are the left N-subgroups of N whereas the second one is the only non-zero proper left ideal of N. Thus each of them is weakly essential and they are essentials too. //

This, in turn, satisfies the hypothesis in 3.2.15. So we re-state the above result in

3.2.17. Theorem : If in a strongly semiprime strictly left Goldie near-ring N, every weakly essential left N-subgroup of N is also essential, then N satisfies the dcc on left annihilators. /

3.3. Maximal left annihilators

3.3.1. Proposition : Let N be with acc on left annihilators and is such that $P = l(A)$ is a maximal left annihilator for some (non-zero) left N-subset A of N. Then P is a strongly prime ideal of N.

Proof : P being a left annihilator of a left N-subset of N, by 1.3.6., it follows that P is an ideal of N.

Let I, J be two invariant subsets of N such that $IJ \subseteq P$.

Then $IJr(P) \subseteq Pr(P) = (0)$

$\Rightarrow \quad IJr(P) = (0)$

$\Rightarrow \quad I \subseteq l(Jr(P))$

Now $Jr(P) = (0)$

$\Rightarrow \quad J \subseteq l(r(P)) = l(r(l(A))) = l(A) = P$.

Again, suppose $Jr(P) \neq (0)$.

Since $r(P)$ is a left N-subset of N (by 1.3.9), it follows that $Jr(P) \subseteq r(P)$.

And so, $l(Jr(P)) \supseteq l(r(P)) = P$.

As $Jr(P) \neq (0)$ and P is maximal with this character, it follows therefore that $P = l(Jr(P))$ and hence $I \subseteq l(Jr(P)) \Rightarrow I \subseteq P$.

Thus we get, in any case, $IJ \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$.

Hence P is a strongly prime ideal of N. //

3.3.2. Proposition [[37]] : Let N be with acc on left annihilators and P_i, P_j be two distinct maximal left annihilators of the type $l(A)$ where A ($\neq 0$) is a left N-subset of N. Then $r(P_i) \subseteq P_j$; $l(P_i) \subseteq P_j$ ($i \neq j$).

Proof : Clearly $P_i\, r(P_i) = (0) \subseteq P_j$.

Since P_i and P_j are ideals of N, these are invariant subsets of N and $r(P_i)$ is also so.

Now P_j being strongly prime, $P_i r(P_i) \subseteq P_j$

$$\Rightarrow P_i \subseteq P_j \text{ or } r(P_i) \subseteq P_j$$

But P_j being maximal and $P_i \neq P_j$, $P_i \subseteq P_j$ is not possible. So, $r(P_i) \subseteq P_j$. Similarly $l(P_i) \subseteq P_j$. //

3.3.3. Proposition[[37]]: Let N be with acc on left annihilators and $P_i = l(A_i)$, $P_j = l(A_j)$ are two distinct maximal left annihilators where A_i, A_j are non-zero left N-subsets of N. Then $A_i \cap A_j = (0)$.

<u>Proof</u> : Here $A_i \cap A_j \subseteq A_i, A_j$

$$\Rightarrow l(A_i \cap A_j) \supseteq l(A_i), l(A_j)$$

$$\Rightarrow l(A_i \cap A_j) \supseteq P_i, P_j$$

$$\Rightarrow P_i = l(A_i \cap A_j) = P_j$$ (for $A_i \cap A_j$ is a left N-subset of N) as P_i, P_j are maximals with this character.

Hence this is possible only when $A_i \cap A_j = (0)$. //

In case of an ideal P of N we get a natural epimorphism

$$\nu : N \to N/_P,\ n \to n + P.$$

Also we know that a left (right or invariant) $N/_P$- subset of $N/_P$ is of the form $K/_P$ where K is a left (right or invariant) N-subset of N such that $P \subseteq K \subseteq N$.

3.3.4. <u>Proposition</u> : Let N be with acc on left annihilators and P is a maximal left annihilator of a non-zero left N-subset of N. Then $N/_P$ is a strongly prime near-ring.

<u>Proof</u> : Let $H/_P$, $K/_P$ be two invariant subsets of $\overline{N}$ $(= N/_P)$ such that $(H/_P)(K/_P) = P$.

Also, $(H/_P)(K/_P) = (HK)/_P$ for $(h + P)(k + P) = hk + P$ when

$h \in H$, $k \in K$.

Thus, $(H/P)(K/P) = P \Rightarrow (HK)/P = P$

$$\Rightarrow HK \subseteq P$$

$$\Rightarrow H \subseteq P \text{ or } K \subseteq P \text{ (as } P \text{ is strongly prime)}$$

$$\Rightarrow H/P = P \text{ or } K/P = P$$

Hence N/P is strongly prime. //

3.3.5. <u>Proposition</u> [[37]] : If P is a maximal left annihilator in a near-ring N with acc on left annihilators and $\bar{J}$ is a left annihilator of a subset of $\bar{N}$ ($= N/P$) then there is a left annihilator J of a subset of N such that $\bar{J} = J/P$.

<u>Proof</u> : Let $\bar{J} = l(T/P)$, where T/P is a subset of N/P.

Define $J = \{x \in N \mid xT \subseteq P\}$ ($\neq \emptyset$ for $0 \in J$)

Thus $JT \subseteq P \Rightarrow J\,Tr(P) \subseteq Pr(P) = (0)$

$$\Rightarrow J \subseteq l(Tr(P))$$

Now $\alpha \in l(Tr(P)) \Rightarrow \alpha Tr(P) = (0)$

$$\Rightarrow \alpha T \subseteq l(r(P)) = P$$

$$\Rightarrow \alpha \in J, \text{ (by definition of } J)$$

$$\Rightarrow l(Tr(P)) \subseteq J.$$

Hence $J = l(Tr(P))$

But $\bar{J} = l(T/_P) = \{x + P \mid (x+P)(t+P) = P,\ x \in N,\ t \in T\}$

$= \{x + P \mid xt \in P\}$

$= \{x+P \mid xT \subseteq P\}$

$= \{x+P \mid x \in J\}$

$= J/_P$.

Thus $\bar{J} = J/_P$ where J is a left annihilator of a subset of N. //

3.3.6. Note : We write $\mathcal{P} = \{ P \mid P$ is a maximal left annihilator of the type $l(A)$, where A is a non-zero left N-subset of N $\}$.

In what follows, we confine our discussion to the subfamily $\Gamma\ (\subseteq \mathcal{P}) = \{ P \mid P = l(A)$, where A is a non-zero invariant subnear-ring of N $\}$.

Now, we furnish an example of such a family $\mathcal{P}$ along with its subfamily Γ in the following .

3.3.7. Example (E(7), Page 339-340 [42]) : $N = \{0,a,b,c\}$ is a near-ring under addition [defined in table 1.1(i)] and multiplication defined by the following table.

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Table : 3.4.

Here non-zero left N subsets are $\{a\}$, $\{b\}$, $\{0,a\}$, $\{0,b\}$, $\{a,b\}$, $\{0,a,b\}$ and N. Clearly their left annihilators are $\{0\}$, $\{0,a\}$ and $\{0,b\}$. Thus $\mathcal{P} = \{\{0,a\}, \{0,b\}\}$.

On the otherhand, non-zero invariant subnear-rings are $\{0,a\}$, $\{0,b\}$ and N and their left annihilators are $\{0\}$, $\{0,a\}$, $\{0,b\}$. Thus, $\Gamma = \{\{0,a\}, \{0,b\}\}$.

Therefore, $\Gamma \subseteq \mathcal{P}$. //

3.4. Left Goldie near-rings.

3.4.1. Theorem : Let N be a strongly semi-prime strictly left Goldie near-ring and Γ ($\subseteq \mathcal{P}$) is the collection of all maximal left annihilators of the type $l(A)$ where A ($\neq 0$) is an invariant subnear-ring of N. Then

(a) Γ is a finite set and

(b) $\bigcap_{P \in \Gamma} P = (0)$.

Proof (a) : Let $\{P_k\}$ be a collection of distinct elements of Γ where $P_k = l(A_k)$ where A_k ($\neq 0$) is an invariant subnear-ring of N.

$$\text{Now } A_k \subseteq r(P_k) \qquad \ldots \qquad \text{(i)}$$

Suppose, $\sum_{k\neq i}^{t} a_k \in r(P_i) \cap \sum_{k\neq i}^{t} r(P_k)$, $a_k \in r(P_k)$, $1 \leq i \leq t$.

and let $\beta_i \in r(P_i)$, then

$a_k\beta_i \in r(P_k)\, r(P_i) \subseteq P_i\, r\,(P_i)$, (by 3.3.2)

$= (0)$

$\Rightarrow\ a_k\beta_i = 0$ (for $k \neq i$)

$\Rightarrow\ (\sum_{k\neq i} a_k)\ \beta_i = 0$

$\Rightarrow\ (\sum_{k\neq i} r(P_k))r(P_i) = 0$ ... (ii)

Now $(r(P_i) \cap \sum_{k\neq i} r(P_k))^2$

$= (r(P_i) \cap \sum_{k\neq i} r(P_k))(r(P_i) \cap \sum_{k\neq i} r(P_k))$

$\subseteq (\sum_{k\neq i} r(P_k))\ r(P_i)$

$= (0)$, [by (ii)]

Thus, $(A_i \cap \sum_{k\neq i} A_k)^2 = (0)$, [by (i)]

And each A_j $(j = 1,2,\ldots,t)$ being an invariant subnear-ring, it is clear that $A_i \cap \sum_{k\neq i} A_k$ is a right N-subset of N and thus by 3.2.5, we get

$A_i \cap \sum_{k\neq i} A_k = (0)$ for N is strongly semiprime.

Therefore, a collection $\{P_k \mid P_k \in \Gamma\}$ gives us an independent family $\{A_k\}$ of left N-subgroups of N. Since, because of strictly left Goldie character, such an independent family of left N-subgroups of N cannot be infinite, Γ must be a finite collection.

(b) Suppose, $\Gamma = \{P_1, P_2, \ldots, P_t\}$ and $X = \bigcap_{k=1}^{t} P_k$.

If $x \in X$ then $x \in P_k$ (for all k)

Also let $\sum_{k=1}^{t} a_k \in \sum_{k=1}^{t} l(P_k)$, where $a_k \in l(P_k)$.

$$\text{Then, } \left(\sum_{k=1}^{t} a_k\right) x = \sum_{k=1}^{t} a_k x \in \sum_{k=1}^{t} l(P_k)\, P_k = (0) .$$

$$\text{Thus, } \left(\sum_{k=1}^{t} l(P_k)\right) X = (0)$$

$$\Rightarrow \sum_{k=1}^{t} l(P_k) \subseteq l(X) \qquad \ldots \qquad (i)$$

Now each of P_k, being an ideal, is an invariant subnear-ring and so is also X. Therefore $l(X)$ must be contained in some (say) $P_m \in \Gamma$ if $X \neq (0)$.

$$\text{So } \sum_{k=1}^{t} l(P_k) \subseteq l(X) \subseteq P_m$$

$$\Rightarrow l(P_m)\left(\sum_{k=1}^{t} l(P_k)\right) \subseteq l(P_m)\, P_m = (0) .$$

$$\text{Thus, } (l(P_m))^2 = l(P_m)\, l(P_m) \subseteq l(P_m) \sum_{k=1}^{t} l(P_k) = (0) .$$

$$\Rightarrow (l(P_m))^2 = (0) .$$

Since $l(P_m)$ is a nilpotent left N-subset of N, it therefore follows by 3.2.5. that $l(P_m) = (0)$, for N is strongly semiprime.

We write $P_m = l(A_m)$ where A_m is a non-zero invariant subnear-ring of N. P_m being an invariant subset of N, by 3.2.6. we get $l(P_m) = r(P_m)$.

So, $r(P_m) = (0)$

$\Rightarrow l(r(l(A_m))) = l(0) = N.$

$\Rightarrow l(A_m) = N$

$\Rightarrow N\ A_m = (0)$

$\Rightarrow A_m = (0)$, not true

Hence $X = (0)$. In otherwords, $\bigcap_{k=1}^{t} P_k = (0)$. //

3.4.2. <u>Theorem</u> [[37]] : Let N be strongly semiprime strictly left Goldie. Then $\overline{N}$ ($= N/_P$ where $P \in \Gamma$) is a strongly prime strictly left Goldie near-ring.

<u>Proof</u> : By 3.3.4, we have that $\overline{N}$ is strongly prime.

And if $\overline{J}_1 \subseteq \overline{J}_2 \subseteq \cdots$ is an ascending chain of left annihilators in $\overline{N}$, by 3.3.5, we get a family $\{J_1, J_2, \ldots\}$ of left annihilators in N such that $J_i = l(T_i r(P))$, $\overline{J}_i = l({}^{T_i}/_P)$ where

${}^{T_i}/_P = \{t_i + P \mid t_i \in T_i\}$ is a subset of $N/_P$.

Now, $x \in J_1 = l(T_1 r(P))$

$$\Rightarrow \quad xT_1 r(P) = (0)$$

$$\Rightarrow \quad xT_1 \subseteq l(r(P)) = P$$

$$\Rightarrow \quad xT_1 \subseteq P$$

$$\Rightarrow \quad xt_1 + P = P, \text{ for } t_1 \in T_1$$

$$\Rightarrow \quad (x+P)(t_1 + P) = P$$

$$\Rightarrow \quad \bar{x}\,\bar{t}_1 = 0$$

$$\Rightarrow \quad \bar{x}\,\bar{T}_1 = \overline{(0)}$$

$$\Rightarrow \quad \bar{x} \in l(\bar{T}_1) = \bar{J}_1 \subseteq \bar{J}_2 \,.$$

$$\Rightarrow \quad \bar{x}\,\bar{T}_2 = \overline{(0)}$$

$$\Rightarrow \quad \overline{x\,T_2} = \overline{(0)}$$

$$\Rightarrow \quad x\,T_2 \subseteq P$$

$$\Rightarrow \quad x\,T_2\,r(P) = (0)$$

$$\Rightarrow \quad x \in l(T_2 r(P)) = J_2$$

$$\Rightarrow \quad J_1 \subseteq J_2$$

Similarly, we get

$J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots$ an ascending chain of left annihilators in N.

Because of Goldie character of N, we get an $s \in Z^{+}$ such that $J_s = J_{s+1} = \cdots$

Now, $\bar{y} \in \bar{J}_{s+1} = l(\bar{T}_{s+1})$

$\Rightarrow \quad \bar{y}\,\bar{T}_{s+1} = \overline{(0)}$

$\Rightarrow \quad \overline{y\,T_{s+1}} = \overline{(0)}$

$\Rightarrow \quad y\,T_{s+1} \subseteq P$

$\Rightarrow \quad y\,T_{s+1}r(P) = (0)$

$\Rightarrow \quad y \in l(T_{s+1}\, r(P)) = J_{s+1} = J_s$.

$\Rightarrow \quad y \in l(T_s r(P))$

$\Rightarrow \quad y\,T_s r(P) = (0)$

$\Rightarrow \quad y\,T_s \in l(r(P)) = P$

$\Rightarrow \quad \bar{y}\,\bar{T}_s = \overline{(0)}$

$\Rightarrow \quad \bar{y} \in l(\bar{T}_s) = \bar{J}_s$

Thus $\bar{J}_{s+1} \subseteq \bar{J}_s \Rightarrow \bar{J}_s = \bar{J}_{s+1}$

Similarly, $\bar{J}_{s+1} = \bar{J}_{s+2} = \cdots$

Therefore, $\bar{N}$ satisfies the acc on left annihilators.

Now, let $\{\bar{J}_1, \bar{J}_2, \ldots, \bar{J}_t\}$ be an independent family of non-zero left $\bar{N}$-subgroups of $\bar{N}$ where each $\bar{J}_i = J_i/P$, J_i ($\supseteq P$) is a left N-subgroup of N and $P = l(A)$, A is an invariant subnear-ring of N. Then $A \cap J_i$ is a left N-subgroup of N. Thus we get a family $\{I_i\}$ ($I_i = A \cap J_i$) of left N-subgroups of N.

We note that $I_i \neq (0)$; for $I_i = (0)$ gives

$AJ_i \subseteq A \cap J_i = I_i = (0)$ (as A is an invariant subnear-ring and J_i is a left N-subgroup of N).

Thus, $(J_iA)^2 = (J_iA)(J_iA) = J_i(AJ_i)A = (0)$.

And therefore $J_iA = (0)$, for N is strongly semi-prime and J_iA is a left N-subset of N.

Therefore, $J_i \subseteq l(A) = P$

$$\Rightarrow \bar{J}_i = \overline{(0)} \text{ , a contradiction.}$$

So, $I_i \neq (0)$.

Now, we prove that $\{I_i\}$ is an independent family of non-zero left N-subgroups of N.

Here, $I_j \cap (\sum_{k\neq j} I_k) = (A \cap J_j) \cap \sum_{k\neq j} (A \cap J_k)$

$$\subseteq A \cap J_j \cap \sum_{k\neq j} J_k$$

Let $\alpha \in J_j \cap \sum_{k\neq j} J_k$, then $\alpha = \alpha_j = \sum_{k\neq j} \alpha_k$, where $\alpha_i \in J_i$ for each i.

Thus $\bar{\alpha} = \bar{\alpha}_j = \sum_{k\neq j} \bar{\alpha}_k$.

$\Rightarrow \bar{\alpha} \in \bar{J}_j \cap \sum_{k \neq j} \bar{J}_k = (\bar{0})$.

$\Rightarrow \bar{\alpha} = \bar{0}$

$\Rightarrow \alpha \in P$

$\Rightarrow J_j \cap \sum_{k \neq j} J_k \subseteq P$

So, $I_j \cap (\sum_{k \neq j} I_k) \subseteq A \cap P \subseteq r(P) \cap P$ (as $A \subseteq r(P)$)

Again, $(r(P) \cap P)^2 \subseteq Pr(P) = (0)$

$\Rightarrow r(P) \cap P = (0)$, by 3.2.5 for $r(P) \cap P$ is a right N-subset of N.

Thus, $I_j \cap (\sum_{k \neq j} I_k) = (0)$.

Hence $\{I_i\}$ is an independent family of left N-subgroups of N.

So, if $\{\bar{J}_1, \bar{J}_2, \ldots\}$ is an infinite independent family of left $\bar{N}$-subgroups of $\bar{N}$, we get an infinite independent family $\{I_i\}$ of left N-subgroups of N and this contradicts the strictly Goldie character of N. Hence $\bar{N}$ cannot have an infinite independent family of left $\bar{N}$-subgroups of $\bar{N}$.

Thus, together with 3.3.4, we get $\bar{N}$ is a strictly left Goldie strongly prime near-ring. //

3.4.3. <u>Theorem</u> : If N is a strongly semiprime strictly left Goldie near-ring and $P_i \in \Gamma$ $(i = 1,2,\ldots t)$ then N is embedded in $N/P_1 \oplus N/P_2 \oplus \cdots \oplus N/P_t$.

<u>Proof</u> : Consider the mapping

$$f : N \rightarrow \bigoplus_{i=1}^{t} N/P_i \quad \text{such that}$$

$$f(n) = (n+P_1, n+P_2), \ldots, n+P_t) \text{ for } n \in N.$$

It is easy to see that for n,m ε N, f(n+m) = f(n)+f(m) and f(nm) = f(n) f(m). Thus f is a near-ring homomorphism.

$$\text{Now, Ker } f = \{n \in N \mid n + P_i = P_i \text{ for all } i\}$$

$$= \{n \in N \mid n \in \cap P_i = (0)\}$$

$$= (0)$$

Hence f is a monomorphism. And therefore N is embedded in $\bigoplus_{i=1}^{t} N/P_i$. //

3.4.4. <u>Example</u> (E(11), Page 339-340 [42]):

Consider the near-ring N = {0,a,b,c} where operation + (defined in the table 1.1(i)) and . is defined as follows.

.	0	a	b	c
0	0	0	0	0
a	0	a	b	a
b	0	0	0	0
c	0	a	b	a

Table : 3.5

Here the non-zero left N-subgroups of N are $\{0,a\}$, $\{0,b\}$ and N where $\{0,b\}$ is its only non-zero proper left ideal. And $a(\neq 0)$ in $\{0,a\}$ is not a non-zero-divisor. Moreover $\{0,a\}$ is not weakly essential.

On the otherhand, $Z_1(N) = (0)$, in otherwords N is left non singular and all the left annihilators (viz. $\{0,b\}$) are distributively generated. Again, $\{0,b\}$ is weakly essential yet b is not a non-zero-divisor.

In the following theorem we see how semiprimeness of N together with distributively generated left annihilators play key role for the existence of a non-zero-divisor in a weakly essential left N-subgroup of N in some special cases.

3.4.5. <u>Theorem</u> [37] : If N is a strongly semiprime strictly left Goldie near-ring with distributively generated left annihilators and weakly essential left N-subgroups of N are essential left N-subgroups then every essential left N-subgroup of N has a non-zero-divisor .

<u>Proof</u> : Let I $(\neq 0)$ be an essential left N-subgroup of N. N being strongly semiprime and with acc on left annihilators, by 3.2.7, I is not nil.

We now consider $a_1 \in I$ with a_1 non nilpotent such that $l(a_1)$ is as large as possible.

If $l(a_1) = (0)$, we stop. If not, then $l(a_1) \cap I \neq (0)$ as $I \subseteq_e N$. Now $l(a_1) \cap I$ being a left N-subgroup of N, it is again non nil.

As above, we choose $a_2 \in l(a_1) \cap I$ with a_2 non nilpotent such that $l(a_2)$ is as large as possible.

Thus $a_2 \in l(a_1) \Rightarrow a_2 a_1 = 0$

Now, $a_1, a_2 \in I \Rightarrow a_1 + a_2 \in I$. If $l(a_1 + a_2) = (0)$, we stop. If not then we get $l(a_1 + a_2) \cap I \neq (0)$. Also $l(a_1 + a_2) \cap I$ is non nil.

First we prove, $l(a_1 + a_2) = l(a_1) \cap l(a_2)$.

For this, let $x \in l(a_1) \cap l(a_2)$ then

$x = \sum_{fin} s_i$ where $s_i \in S_1$, $l(a_1) = \langle S_1 \rangle$, S_1 is a set of distributive elements.

Thus each $s_i \in l(a_1)$. So $s_i a_1 = 0$ (for each i)

$$\Rightarrow (\Sigma s_i)(a_1 + a_2) = \Sigma s_i a_2$$

$$= (\Sigma s_i) a_2$$

$$= x a_2$$

$$= 0$$

$$\Rightarrow x \in l(a_1 + a_2)$$

$$\Rightarrow l(a_1) \cap l(a_2) \subseteq l(a_1 + a_2) \qquad \ldots \qquad (i)$$

Conversely, let $y = \sum_{fin} \alpha_i \in l(a_1 + a_2) = \langle S \rangle$, $\alpha_i \in S$, (a set of distributive elements).

$\Rightarrow \alpha_i \in l(a_1 + a_2)$, for each i

$\Rightarrow \alpha_i (a_1 + a_2) = 0$

$\Rightarrow \alpha_i a_1 + \alpha_i a_2 = 0$

$\Rightarrow \alpha_i a_1^2 + \alpha_i a_2 a_1 = 0$

$\Rightarrow \alpha_i a_1^2 = 0$, (as $a_2 a_1 = 0$)

$\Rightarrow \alpha_i \in l(a_1^2) \supseteq l(a_1)$

$\Rightarrow \alpha_i \in l(a_1)$ [for $l(a_1)$ being maximal, $l(a_1^2) = l(a_1)$]

$\Rightarrow \alpha_i a_1 = 0$ (for each i)

and so, $\alpha_i a_2 = 0$ giving thereby

$(\sum \alpha_i) a_1 = 0$ and $(\sum \alpha_i) a_2 = 0$

$\Rightarrow y = \sum \alpha_i \in l(a_1) \cap l(a_2)$

$\Rightarrow l(a_1 + a_2) \subseteq l(a_1) \cap l(a_2)$ ... (ii)

Thus, (i) and (ii) give, $l(a_1 + a_2) = l(a_1) \cap l(a_2)$ .. (iii)

Since $l(a_1 + a_2) \cap I = l(a_1) \cap l(a_2) \cap I$ is a non-zero non nil left N-subgroup, we choose $a_3 \in l(a_1) \cap l(a_2) \cap I$ with a_3 non nilpotent such that $l(a_3)$ is as large as possible.

Now $a_1 + a_2 + a_3 \in I$. If $l(a_1 + a_2 + a_3) = 0$ we stop. And if not, we proceed as above and as a result we get

$$l(a_1) \supseteq l(a_1) \cap l(a_2) \supseteq l(a_1) \cap l(a_2) \cap l(a_3) \supseteq \ldots$$

$$\supseteq l(a_1) \cap \ldots \cap l(a_t) \supseteq \ldots$$ (for each t, we have

$l(a_1) \cap l(a_2) \cap \ldots \cap l(a_t) = l(a_1 + a_2 + \ldots + a_t)$ as above).

Since N satisfies the conditions of 3.2.17, we get $t \in Z^+$ such that

$$l(a_1 + a_2 + \ldots + a_t) = l(a_1 + a_2 + \ldots + a_t + a_{t+1})$$

$$= l(a_1 + a_2 + \ldots + a_t) \cap l(a_{t+1})$$

$$\Rightarrow \quad l(a_1 + a_2 + \ldots + a_t) \subseteq l(a_{t+1})$$

But by our choice, $a_{t+1} \in l(a_1 + a_2 + \ldots + a_t) \cap I$

Thus, $a_{t+1} \in l(a_1 + a_2 + \ldots + a_t) \Rightarrow a_{t+1} \in l(a_{t+1})$

$$\Rightarrow a_{t+1}^2 = 0$$

And this contradicts with our choice of a_{t+1} (non nilpotent).

So, $a_{t+1} = 0$. Therefore, $l(a_1 + a_2 + \ldots + a_t) \cap I = (0)$ giving thereby $l(a_1 + a_2 + \ldots + a_t) = (0)$ (as $I \subseteq_e N$).

Thus we get $c_1 = a_1 + a_2 + \ldots + a_t \in I$ such that $l(c) = (0)$

And by 3.2.9, it follows that $Nc \subseteq_e N$ and 3.2.14 in turn, gives that c is a non-zero-divisor. //

3.5. Near-rings of left quotients of a left Goldie near-ring

3.5.1. Lemma : If N satisfies the dcc on its left N-subgroups and non nilpotent elements of N are distributives, then every non nil left N-subgroup I of N contains a non-zero idempotent.

Proof : Let $\mathfrak{Y} = \{ J \subseteq I \mid J$ is a non nil left N-subgroup of N$\}$.

Clearly $\mathfrak{Y} \neq \emptyset$, as $I \in \mathfrak{Y}$. Since N satisfies the dcc on left N-subgroups, $\mathfrak{Y}$ has a minimal element (say) H. Now H being non nil, H^2 is also non nil. But $H^2 \subseteq H$, hence $H^2 = H$ as H is minimal.

Now, consider the family $C = \{L \subseteq H \mid L$ is a non nil left N-subgroup of N, $HL \neq (0)\}$.

Here it is easy to see that $C \neq \emptyset$ for $H \in C$ (as $H^2 = H \neq (0)$). So C has a minimal element (say) L_1. Thus $HL_1 \neq (0)$, $L_1 \subseteq H$ and L_1 is non nil.

Let $u\ (\neq 0) \in L_1$ such that u is non nilpotent.

Then $Hu \neq (0)$ and $Hu \subseteq L_1$. Therefore we have $a \in H$ such that $au = u$. So, $a^t u = a^{t-1}(au) = a^{t-1}u = \ldots = u$ for all $t \in Z^+$.

Since $u \neq 0$, $a^t \neq 0$ for all $t \in Z^+$. Hence a is not nilpotent and therefore, by assumption, it is distributive.

Again $l_H(u)$ is a left N-subgroup of N and also, $l_H(u) \subseteq H$.

Now, $u \in l_H(u) \Rightarrow u^2 = 0$, not true.

So, $u \notin l_H(u)$. Hence $l_H(u) \subset H$. And the minimality of H in $\mathcal{H}$ therefore gives that $l_H(u)$ is nil.

Also, $(a^2 - a)\, u = a^2 u - au = au - au = 0$

$$\Rightarrow \quad a^2 - a \in l_H(u) \quad (\text{for } a \in H)$$

Thus, $a^2 - a$ is nilpotent.

Let $(a^2 - a)^t = 0$ for some $t \in Z^+$

As a is distributive, we therefore get from above

$a^t = a^{t+1}\, g(a)$, where $g(x)$ is a polynomial in x with coefficient +1 or -1 . Again a being distributive, a^t is also so. Therefore, we get

$$a^t\, g(a) = g(a)\, a^t$$

Now, $a^t = a^{t+1}\, g(a) = a^{t+2}\, (g(a))^2 = \ldots$

$$= a^{t+t}\, (g(a))^t = a^t (a^t (g(a))^t) = a^t e$$

where $e = a^t (g(a))^t = (g(a))^t a^t$.

Therefore, $e^2 = a^t (g(a))^t\, a^t (g(a))^t$

$$= a^t e\, (g(a))^t$$

$$= a^t (g(a))^t$$

$$= e$$

If $e = 0$ then $a^t = a^t e = 0$, a contradiction.

Thus e is a non-zero idempotent.

Again, $a \in H \subseteq I \Rightarrow g(a) \in I$

and so a^t, $(g(a))^t \in I$

Hence $e = a^t(g(a))^t \in I$

Therefore I contains a non-zero idempotent. //

3.5.2. <u>Lemma</u> : Let N satisfy the dcc on its left N-subgroups and non nilpotent elements are distributives. If N has no non-zero nil left N-subgroup of N then every left N-subgroup I ($\neq$ 0) of N contains a non-zero idempotent e such that I = Ne.

<u>Proof</u> : Since I is non nil, by 3.5.1, I contains an idempotent e' (say).

Now the family $\{l_I(e'') \mid e'' \text{ is an idempotent of } I\}$ of left N-subgroups of N has a minimal element $l_I(e)$, (say).

Again, $l_I(e) \neq (0) \Rightarrow l_I(e)$ contains an idempotent (say) e_1 (by 3.5.1). Then $e_1e = 0$.

Each of e and e_1 being idempotent, is non nilpotent and therefore distributive.

Write, $e_2 = e + e_1 - ee_1 \in I$ (as e, $e_1 \in I$)

Then $e_2e = (e + e_1 - ee_1)\, e = e$

$\Rightarrow \quad e_2^2 = (e+e_1-ee_1)(e+e_1 - ee_1)$

$= e(e+e_1- ee_1) + e_1(e+e_1 - ee_1) - ee_1 (e+e_1 - ee_1)$

$= e + e_1 - ee_1 \quad (\text{as } e_1e = 0,\ e_1^2 = e_1,\ e^2 = e)$

$= e_2$

So, e_2 is an idempotent such that $e_2 \varepsilon I$.

Now, $x \varepsilon l_I(e_2)$

$\Rightarrow \quad xe_2 = 0$

$\Rightarrow \quad x(e_2e) = 0$

$\Rightarrow \quad xe = 0$

$\Rightarrow \quad x \varepsilon l_I(e)$.

Thus, $l_I(e_2) \subseteq l_I(e)$

Again, $e_1e_2 = 0 \Rightarrow e_1 \varepsilon l_I(e_2)$

$\Rightarrow e_1 \varepsilon l_I(e+e_1 - ee_1)$

$\Rightarrow e_1 (e + e_1 - ee_1) = 0$

$\Rightarrow e_1 = 0 \quad (\text{as } e_1^2 = e_1,\ e_1e = 0)$

Hence $e_1e_2 = 0$ gives rise a contradiction.

Therefore, $e_1 \not\varepsilon l_I(e_2)$. But $e_1e = 0$ gives that $e_1 \varepsilon l_I(e)$. Thus $l_I(e_2) \subset l_I(e)$, which contradicts the minimality of $l_I(e)$, we therefore get $l_I(e) = (0)$.

Now, for $y \in I$, $(y - ye)\, e = ye - ye = 0$

$$\Rightarrow \; y - ye \in l_I(e) = (0)$$

$$\Rightarrow \; y = ye \in Ne$$

$$\Rightarrow \; I \subseteq Ne \, .$$

But $Ne \subseteq I$ (as $e \in I$ and I is a left N-subgroup of N).

Hence $I = Ne$. //

3.5.3. Definition : Let S be a multiplicative subsemigroup of N. Then N is said to satisfy (left) Ore condition w.r.t. S if for each pair $(s,n) \in S \times N$ there exists $(s_1,n_1) \in S \times N$ such that $s_1 n = n_1 s$.

3.5.4. Definition : A left N-subgroup D of N is said to be dense if for all $n \in N$, $nD = (0)$ implies $n = 0$.

3.5.5. Definitions : Let N be a subnear-ring of a near-ring Q. Q is said to be a near-ring of left quotients of N if for all $q \in Q$, $N q^{-1}$ ($= \{x \in Q \mid xq \in N\}$) is a dense left N-subgroup of Q.

A (right) near-ring $Q(N)$ of left quotients of N is called a complete near-ring of left quotients of N if a monomorphism : $N \to Q(N)$ can be extended to a monomorphism : $Q \to Q(N)$ where Q is a near-ring of left quotients of N.

A right near-ring C_{cl} containing N as a subnear-ring is

Now, for $y \in I$, $(y - ye)\, e = ye - ye = 0$

$$\Rightarrow \; y - ye \in l_I(e) = (0)$$

$$\Rightarrow \; y = ye \in Ne$$

$$\Rightarrow \; I \subseteq Ne \, .$$

But $Ne \subseteq I$ (as $e \in I$ and I is a left N-subgroup of N).

Hence $I = Ne$. //

3.5.3. Definition : Let S be a multiplicative subsemigroup of N. Then N is said to satisfy (left) Ore condition w.r.t. S if for each pair $(s,n) \in S \times N$ there exists $(s_1,n_1) \in S \times N$ such that $s_1 n = n_1 s$.

3.5.4. Definition : A left N-subgroup D of N is said to be dense if for all $n \in N$, $nD = (0)$ implies $n = 0$.

3.5.5. Definitions : Let N be a subnear-ring of a near-ring Q. Q is said to be a near-ring of left quotients of N if for all $q \in Q$, $N q^{-1}$ ($= \{x \in Q \mid xq \in N\}$) is a dense left N-subgroup of Q.

A (right) near-ring $Q(N)$ of left quotients of N is called a complete near-ring of left quotients of N if a monomorphism : $N \to Q(N)$ can be extended to a monomorphism : $Q \to Q(N)$ where Q is a near-ring of left quotients of N.

A right near-ring C_{cl} containing N as a subnear-ring is called a <u>classical near-ring of left quotients</u> of N w.r.t. a semigroup S of distributive non-zero-divisors of N if and only if

(i) $1 \in C_{cl}$

(ii) elements of S are invertible in C_{cl} and

(iii) for each $x \in C_{cl}$ there exists $s \in S$ such that $sx \in N$.

3.5.6. <u>Note</u> 1. In what follows we shall write Q to denote the classical near-ring C_{cl} of left quotients of N w.r.t. S.

2. $Q = \{s^{-1}n \mid s \in S, n \in N\}$.

3.5.7. <u>Lemma</u> : Let Q be a classical near-ring of left quotients of N w.r.t. S (a multiplicative semigroup of distributive non-zero-divisors of N) and N satisfies the (left) Ore condition w.r.t.S.

If $s_1, s_2, \ldots, s_t \in S$ then there exist $n_1, n_2, \ldots, n_t \in N$ and $s \in S$ such that $s_i^{-1} = s^{-1}n_i$ ($i = 1, 2, \ldots, t$).

<u>Proof</u> : If $s = s_1^2$ and $n_1 = s_1$ then $s^{-1}n_1 = s_1^{-1}$ and thus the result is true for $t = 1$.

Assume that the result holds good for $t - 1$. Then for $s_1, s_2, \ldots, s_{t-1} \in S$, we have $m_1, m_2, \ldots, m_{t-1} \in N$, $u \in S$ such that $s_i^{-1} = u^{-1}m_i$, $i = 1, 2, \ldots, t-1$.

Again by (left) Ore condition, for $(u, s_t) \in S \times N$ we have $(n_t, m) \in S \times N$ such that $n_t s_t = mu = s$ (say). Then $s \in S$ and $m(= su^{-1})$ is invertible in Q.

Write $n_i = m\ m_i \quad (i = 1,2,\ldots, t-1)$

$$\Rightarrow \quad m_i = m^{-1} n_i$$

Then $s_i^{-1} = u^{-1} m_i = u^{-1}(m^{-1} n_i) = (u^{-1} m^{-1})\ n_i$

$$= (mu)^{-1} n_i = s^{-1} n_i, \quad \because s = mu \in S, \ (i = 1,2,\ldots t-1)$$

Also, $s_t^{-1} = s^{-1} n_t$

And thus the result follows. //

3.5.8. __Lemma :__ If J is a left N-subgroup of N then the elements of $S^{-1}J \; (= \{\sum_{fin} s_i^{-1} x_i \mid (s_i, x_i) \in S \times J\})$ are of the form $s^{-1} j$ where $s \in S$, $j \in J$.

__Proof :__ Let $\alpha = \sum_{i=1}^{t} s_i^{-1} x_i \in S^{-1}J$.

Since $s_1, s_2, \ldots, s_t \in S$, by 3.5.7 we have $n_1, n_2, \ldots, n_t \in N$, $s \in S$ such that $s_i^{-1} = s^{-1} n_i$. $(i = 1,2,\ldots, t)$

Thus, $\alpha = \sum_{i=1}^{t} s^{-1} n_i x_i$

$$= s^{-1} \sum_{i=1}^{t} n_i x_i \text{ (as s being distributive, } s^{-1} \text{ is also so.)}$$

$= s^{-1} \sum_{i=1}^{t} j_i \quad (j_i = n_i x_i \in J, \text{ for } x_i \in J)$

$= s^{-1} j , \quad (j = \sum_{i=1}^{t} j_i) .$ //

3.5.9. <u>Lemma</u> : If J is a left N-subgroup of N, then $S^{-1}J$ ($= \{s^{-1}x \mid (s,x) \in S \times J\}$) is a left Q-subgroup of Q.

<u>Proof</u> : For, $s_1^{-1} x,\ s_2^{-1} y \in S^{-1}J$,

$s_1^{-1} x - s_2^{-1} y = s_1^{-1} x + s_2^{-1}(-y) \in S^{-1}J$, (from 3.5.8).

Now, let $q \in Q$, then $q = s^{-1}n$, for $s \in S$, $n \in N$.

So, $q(s_1^{-1}x) = (s^{-1}n)(s_1^{-1} x)$

$= ((s^{-1}n)(s^{-1}1))\, x$

$= (s_2^{-1}m)x, \quad [s_2^{-1}m = (s^{-1}n)(s_1^{-1} 1) \in Q]$

$= s_2^{-1} (mx)$

$= s_2^{-1} y$, where $y = mx \in J$ as $x \in J$, $m \in N$

$\Rightarrow \quad q(s_1^{-1} x) = s_2^{-1} y \in S^{-1}J.$

Therefore, $S^{-1}J$ is a left Q-subgroup of Q. //

3.5.10. <u>Lemma</u> : If $J = l(T)$ where $T \subseteq N$ then

(i) $S^{-1}J \cap N = J$

(ii) $S^{-1}J = l_Q(T)$.

<u>Proof</u> : (i) $x \varepsilon J \Rightarrow x \varepsilon S^{-1}J \cap N$ (for $x = 1^{-1}x$)

$$\Rightarrow J \subseteq S^{-1}J \cap N .$$

Conversely, let $y \varepsilon S^{-1}J \cap N$ then

$y = s^{-1}a = n, \quad a \varepsilon J, s \varepsilon S, n \varepsilon N$

$\Rightarrow (s^{-1}a)\ T = (0)$, for $a \varepsilon l\ (T) = J$

$\Rightarrow s^{-1}a \varepsilon l(T) = J$

$\Rightarrow y \varepsilon J$

$\Rightarrow S^{-1}J \cap N \subseteq J.$

Thus, $S^{-1}J \cap N = J.$ //

(ii) Let $x \varepsilon l_Q(T)$ then $xT = (0)$ where $x = s^{-1}n, s \varepsilon S, n \varepsilon N.$

So, $(s^{-1}n)\ T = (0)$

$\Rightarrow s^{-1}(nT) = (0)$

$\Rightarrow nT = (0)$

$\Rightarrow n \varepsilon l(T) = J.$

$\Rightarrow x = s^{-1}n \varepsilon S^{-1}J.$

$\Rightarrow l_Q(T) \subseteq S^{-1}J.$

Conversely, let $y \varepsilon S^{-1}J$ then

$y = s^{-1}j$, for $s \varepsilon S, j \varepsilon J.$

So, $jT = (0)$, as $J = l(T)$

$\Rightarrow (s^{-1}j)\, T = (0)$

$\Rightarrow s^{-1}j \in l_Q(T)$, as $s^{-1}j \in Q$

$\Rightarrow S^{-1}J \subseteq l_Q(T)$

Therefore, $S^{-1}J = l_Q(T)$. //

3.5.11. Lemma : If $\{J_1, J_2, \ldots, J_t\}$ is an independent family of left N-subgroups of N then $\{S^{-1}J_1, S^{-1}J_2, \ldots, S^{-1}J_t\}$ is an independent family of left Q-subgroups of Q.

Proof : By the above lemma each $S^{-1}J_i$ ($i = 1,2,\ldots, t$) is a left Q-subgroup of Q.

Now, let $y \in S^{-1}J_m \cap \sum_{i \neq m}^{t} S^{-1}J_i$, $1 \leq m \leq t$

Then, $y = s_m^{-1} j_m = s_1^{-1} j_1 + \ldots + \widehat{s_m^{-1} j_m} + \ldots + s_t^{-1} j_t$ ...(i)

where $s_1, \ldots, s_t \in S$, $j_1, \ldots, j_t \in J$ and $\wedge$ carries usual meaning of omission .

Now, by 3.5.7, we get $n_1, \ldots, n_t \in N$, $s \in S$ such that

$s_i^{-1} = s^{-1}n_i$ ($i = 1,2,\ldots, t$)

Then by (i), $s^{-1}n_m j_m = s^{-1}n_1 j_1 + \ldots + \widehat{s^{-1}n_m j_m} + \ldots + s^{-1}n_t j_t$

$= s^{-1}(n_1 j_1 + \ldots + \widehat{n_m j_m} + \ldots + n_t j_t)$, as s^{-1} distributive

$\Rightarrow \quad n_m j_m = n_1 j_1 + \ldots + \hat{n_m j_m} + \ldots + n_t j_t \in J_m \cap \sum_{i \neq m}^{t} J_i = (0)$

(Since $\{J_1, \ldots, J_t\}$ is independent)

Thus, $y = s_m^{-1} j_m = s^{-1} n_m j_m = 0$

Hence $S^{-1} J_m \cap \sum_{i \neq m}^{t} S^{-1} J_i = (0)$

Thus $\{ S^{-1} J_1, S^{-1} J_2, \ldots, S^{-1} J_t\}$ is an independent family of left Q-subgroups of Q. //

3.5.12. <u>Lemma</u> : Let A be any subset of a d.g.nr. N then the set $QAQ = \{\sum_{fin} x_i a_i y_i \mid x_i, y_i \in Q,\ a_i \in A\}$ is a left Q-subgroup of Q.

<u>Proof</u> : For any $\alpha = \sum_{fin} x_i a_i y_i$, $\beta = \sum_{fin} \gamma_i b_i \delta_i \in QAQ$, we clearly have $\alpha - \beta \in QAQ$ as we can write $-\gamma_i b_i \delta_i = (-\gamma_i) b_i \delta_i$.

Since N is d.g., Q is also so.

Now, let $q \in Q$ then $q = \sum_{j=1}^{t} q_j$, q_i's are distributive elements of Q.

So, $q\alpha = \left(\sum_{j=1}^{t} q_j \right) \left(\sum_{i=1}^{r} x_i a_i y_i \right)$

$= \sum_{i=1}^{r} (q_1 x_i)\, a_i y_i + \ldots + \sum_{i=1}^{r} (q_t x_i)\, a_i y_i \in QAQ$.

Therefore, QAQ is a left Q-subgroup of Q. //

For the sake of completeness, we describe the results of Tiwari and Seth [53] leading to the sufficient condition for coincidence of a complete near-ring Q(N) of left quotients with a classical near-ring of left quotients of N.

Here N is a d.g.nr. with 1 and S is a semigroup of some distributive non-zero-divisors of N. Here N satisfies the common left multiple property (CLMP), in otherwords the (left) Ore condition w.r.t.S.

3.5.13. Lemma (Lemma 1.1 [53]) : For $\lambda \in S$, $N\lambda$ is a dense left N-subgroup of N.

3.5.14. Lemma (Lemma 1.2 [53]) : Every dense left N-subgroup is weakly essential.

3.5.15. Corollary (Cor. 1.1 [53]) : Q(N) is a regular near-ring.

3.5.16. Lemma (Lemma 1.4 [53]) : If $\lambda \in S$ then λ is a distributive non-zero-divisor in Q(N) .

3.5.17 Lemma (Lemma 1.5[53]) : Every non-zero-divisor of Q(N) is invertible in Q(N).

Now, if $q \in Q(N)$ then Nq^{-1} is dense. Hence it is weakly essential. So it contains a $\lambda \in S$. Thus $\lambda q = a \in N$ or $q = \lambda^{-1} a \in C_{cl}$

Hence Q(N) is a classical near-ring of left quotients w.r.t.S. Thus we get

3.5.18 Lemma (Theorem 1.1[53]): The complete near-ring Q(N) of left quotients of N is a classical near-ring of left quotients w.r.t. S. //

In the remaining part of the section, we consider N with distributively generated left annihilator such that a weakly essential left N-subgroup of N is an essential left N-subgroup; S will mean (as in lemma 3.5.7) a multiplicative semigroup of distributive non-zero-divisors.

3.5.19 Theorem : If N is strongly semiprime strictly left Goldie then N satisfies the Ore condition w.r.t. S.

Proof : Let $a \in S$. Then by 3.2.9, $Na \subseteq_e N$. And so by 3.1.8, $(Na; b) \subseteq_e N$, for $b \in N$.

Therefore, by 3.4.5, $(Na ; b)$ contains a non-zero-divisor (say) a_1.

Thus, $a_1 b = b_1 a$ for some $b_1 \in N$.

Hence N satisfies the Ore condition w.r.t.S. //

Now we note that in case of an N as in theorem 3.5.19, Ore condition is satisfied w.r.t. S and if N is d.g.nr. also with S (a semigroup of distributive non-zero-divisors) then by lemma 3.5.18, we get

3.5.20 Theorem [[37]] : If N is a strongly semiprime strictly left Goldie d.g.nr. then the complete near-ring Q(N) of left quotients of N is a classical near-ring of left quotients of N w.r.t. S. //

3.5.21 Theorem : Let N be a strongly semiprime strictly left Goldie d.g.nr. with Q as classical near-ring of left quotients of N w.r.t. S.

Then Q has no nilpotent left Q-subgroup.

Proof : If possible, let L be a left Q-subgroup of Q such that $L^2 = (0)$. Then $L \cap N$ is a left N-subgroup of N.

Write $S^{-1}(L \cap N) = \{s^{-1}x \mid s \in S, x \in L \cap N\}$.

Let $y \in L$. Then $y = s^{-1}n$ for some $s \in S$, $n \in N$.

So, $sy = n \in N$. Also, $sy \in L$ (as L is a left Q-subgroup of Q).

Therefore, $sy \in L \cap N$

$$\Rightarrow y = s^{-1}(sy) \in S^{-1}(L \cap N)$$

$$\Rightarrow L \subseteq S^{-1}(L \cap N)$$

Again, $S^{-1}(L \cap N) \subseteq S^{-1}L \subseteq L$.

Thus, $L = S^{-1}(L \cap N)$

Also, $(L \cap N)^2 = (L \cap N)(L \cap N) \subseteq LL = L^2 = (0)$

$$\Rightarrow (L \cap N)^2 = (0)$$

So, $L \cap N$ is a nilpotent left N-subgroup of N. And by 3.2.5, we therefore get $L \cap N = (0)$ which gives in turn, $L=S^{-1}(L \cap N)=(0)$.

Thus $L^2 = (0)$ gives $L = (0)$.

Hence Q has no non-zero nilpotent left Q-subgroup. //

In what follows N contains distributive non-zero-divisors only.

3.5.22. Theorem [[37]] : If N is a strongly semiprime strictly left Goldie d.g.nr. then the classical near-ring Q of left quotients of N w.r.t. S satisfies the dcc on its left Q-subgroups.

Proof : Let A,B be two left Q-subgroups of Q such that $B \subset A$.

Now, $B \cap N \subseteq_e A \cap N$ implies $M \subseteq_e N$, for some left N-subgroup M of N with $M\alpha \neq (0)$, $M\alpha \subseteq B \cap N$ for each non-zero $\alpha \in A \cap N$ (by 3.1.6.) .

And therefore by 3.4.5, M contains a non-zero-divisor (say) c.

Thus, $\alpha = c^{-1}(c\alpha) \in Q\,(B \cap N) \subseteq QB \subseteq B$

$\Rightarrow A \cap N \subseteq B$

As in the proof of 3.5.21, we get

$$A = S^{-1}(A \cap N)$$

$\Rightarrow A \subseteq S^{-1}B \subseteq B$, a contradiction.

Therefore, $B \cap N \nsubseteq_e A \cap N$.

So, we have a non-zero left N-subgroup X of $A \cap N$ such that $X \cap (B \cap N) = (0)$.

we therefore get $L \cap N = (0)$ which gives in turn, $L = S^{-1}(L \cap N) = (0)$.

Thus $L^2 = (0)$ gives $L = (0)$.

Hence Q has no non-zero nilpotent left Q-subgroup. //

In what follows N contains distributive non-zero-divisors only.

3.5.22. Theorem [37] : If N is a strongly semiprime strictly left Goldie d.g.nr. then the classical near-ring Q of left quotients of N w.r.t. S satisfies the dcc on its left Q-subgroups.

Proof : Let A,B be two left Q-subgroups of Q such that $B \subset A$.

Now, $B \cap N \subseteq_e A \cap N$ implies $M \subseteq_e N$, for some left N-subgroup M of N with $M\alpha \neq (0)$, $M\alpha \subseteq B \cap N$ for each non-zero $\alpha \in A \cap N$ (by 3.1.6.) .

And therefore by 3.4.5, M contains a non-zero-divisor (say) c.

Thus, $\alpha = c^{-1}(c\alpha) \in Q\ (B \cap N) \subseteq QB \subseteq B$

$\Rightarrow A \cap N \subseteq B$

As in the proof of 3.5.21, we get

$$A = S^{-1}(A \cap N)$$

$\Rightarrow \quad A \subseteq S^{-1}B \subseteq B$, a contradiction.

Therefore, $B \cap N \not\subseteq_e A \cap N$.

So, we have a non-zero left N-subgroup X of $A \cap N$ such that $X \cap (B \cap N) = (0)$.

$\Rightarrow$ $Nx \cap (B \cap N) \subseteq X \cap (B \cap N) = (0)$, (for $x \in X$)

$\Rightarrow$ $Nx \cap (B \cap N) = (0)$, $Nx \subseteq X \subseteq A \cap N$.

Similarly, if C is a left Q-subgroup such that $C \subset B$, then we get a non-zero left N-subgroup $Y \subseteq B \cap N$ and $Ny \cap (C \cap N) = (0)$, (for $y \in Y$), $Ny \subseteq B \cap N$.

Therefore, $Nx \cap Ny \subseteq Nx \cap (B \cap N) = (0)$.

$\Rightarrow$ $Nx \cap Ny = (0)$.

Hence $\{Nx, Ny\}$ is an independent family.

As above, if D is another left Q-subgroup of Q with $D \subset C$, then we get a left N-subgroup Z ($\neq (0)$) of N such that $Z \subseteq C \cap N$, $Nz \cap (D \cap N) = (0)$ for $z \in Z$, $Nz \subseteq C \cap N$.

So, $Nx \cap Nz = (0) = Ny \cap Nz$.

Now, $Nx \cap (Ny + Nz) \subseteq Nx \cap ((B \cap N) + (C \cap N))$

$\subseteq Nx \cap (B \cap N) = (0)$, (as $Ny \subseteq B \cap N$,

$Nz \subseteq C \cap N \subseteq B \cap N$)

$\Rightarrow$ $Nx \cap (Ny + Nz) = (0)$

If $\alpha \in Ny \cap (Nz + Nx)$, then $\alpha = n_1 y = n_2 z + n_3 x$, $(n_1, n_2, n_3 \in N)$

$\Rightarrow$ $-n_3 x = -n_1 y + n_2 z \in Nx \cap (Ny + Nz) = (0)$.

$\Rightarrow$ $n_1 y = n_2 z \in Ny \cap Nz = (0)$

$\Rightarrow$ $\alpha = 0$

Thus, $Ny \cap (Nz + Nx) = (0)$

Similarly, $Nz \cap (Nx + Ny) = (0)$.

Proceeding in like manner, we get an infinite independent family $\{Nx, Ny, Nz, \ldots\}$ of left N-subgroups of N with the strictly descending chain $A \supset B \supset C \supset \ldots$ of left Q-subgroups of Q.

Since N is strictly left Goldie, we cannot get such an infinite independent family of left N-subgroups of N.

Therefore, Q cannot have an strictly descending infinite chain of left Q-subgroups. In otherwords, Q must satisfy the dcc on its left Q-subgroups. //

We now discuss a partial converse of above results.

3.5.23. <u>Theorem</u> : If the classical near-ring Q of left quotients of N is strictly left Goldie then so is also N.

<u>Proof</u> : Let $J_1 \subseteq J_2 \subseteq J_3 \subseteq \ldots$ be an ascending chain of left annihilators such that

$$J_i = l(T_i), \quad T_i \subseteq N.$$

Then $S^{-1}J_1 \subseteq S^{-1}J_2 \subseteq S^{-1}J_3 \subseteq \ldots$ and

each $S^{-1}J_i = l_Q(T_i)$, (by 3.5.10)

As Q is strictly left Goldie, we therefore get $t \in Z^+$ such that $S^{-1}J_t = S^{-1}J_{t+1} = \ldots$

$$\Rightarrow (S^{-1}J_t) \cap N = (S^{-1}J_{t+1}) \cap N = \dots$$

$$\Rightarrow J_t = J_{t+1} = \dots , \quad \text{(by 3.5.10)}$$

Hence N satisfies the acc on left annihilators.

Next, if $\{A_i\}$ is an independent family of left N-subgroups of N, then by 3.5.11, $\{S^{-1}A_i\}$ is an independent family of left Q-subgroups of Q.

So the family $\{A_i\}$ cannot be an infinite one; otherwise $\{S^{-1}A_i\}$ will be an infinite family contradicting the Goldie character of Q.

Thus N is strictly left Goldie. //

3.5.24. __Theorem__ : Let Q (the classical near-ring of left quotients of a d.g.nr. N) possesses only central idempotent satis - fying the dcc on its left Q-subgroups.

If non nilpotent elements of Q are distributives and Q has no non-zero nil left Q-subgroup, then N has no non-zero nilpotent left N-subgroup of it.

__Proof__ : Let K be a nilpotent left N-subgroup of N such that $K^2 = (0)$.

Now, by 3.5.12, Q K Q is a left Q-subgroup of Q. So by 3.5.2, there exists an idempotent e ($\neq 0$) ε Q such that

$QKQ = Qe.$

As $e \in Qe$ we write $e = \sum_{fin} x_i k_i y_i$, $x_i, y_i \in Q$, $k_i \in K$.

Now, $x_i \in Q \Rightarrow x_i = s_i^{-1} n_i$, $s_i \in S$, $n_i \in N$.

By 3.5.7., we have $u_i \in N$, $s \in S$ such that

$$s_i^{-1} = s^{-1} u_i$$

So, $$e = \sum_{fin} x_i k_i y_i = \sum_{fin} s_i^{-1} n_i k_i y_i$$

$$= \sum_{fin} s^{-1} u_i n_i k_i y_i$$

$$= s^{-1} \sum_{fin} u_i n_i k_i y_i$$

$$= s^{-1} \sum_{fin} (u_i n_i k_i) y_i$$

$$= s^{-1} \sum_{fin} v_i y_i, \text{ where } v_i = u_i n_i k_i \in K$$

$$\Rightarrow \quad se = \sum_{fin} v_i y_i \in KQ.$$

$$\Rightarrow \quad K\,se \subseteq K^2 Q = (0)$$

$$\Rightarrow \quad Kse = (0)$$

$$\Rightarrow \quad Kes = (0), \quad \text{(as } e \text{ is central)}$$

$\Rightarrow$ $Ke = (0)$, (as s is non-zero-divisor)

Also, $K \subseteq QKQ = Qe$

Now, $k \in K \Rightarrow k = qe$, $q \in Q$

$$\Rightarrow k = (qe)\, e$$

$$= ke$$

$$\Rightarrow K \subseteq Ke = (0)$$

$$\Rightarrow K = (0)$$

Thus N has no non-zero nilpotent left N-subgroup of it. //

$\Rightarrow$ $Ke = (0)$, (as s is non-zero-divisor)

Also, $K \subseteq QKQ = Qe$

Now, $k \in K \Rightarrow k = qe$, $q \in Q$

$$\Rightarrow k = (qe)\, e$$

$$= ke$$

$$\Rightarrow K \subseteq Ke = (0)$$

$$\Rightarrow K = (0)$$

Thus N has no non-zero nilpotent left N-subgroup of it. //

CHAPTER IV

Some radical characters of a left Goldie near-ring

A subnear-ring of a Goldie near-ring need not be Goldie. But here we prove ([[14]])that some properties of a Goldie near-ring (existence of classical near-ring of left quotients in particular) are inherited by a subnear-ring (without being Goldie) when the parent near-ring is radical over it. Here we extend some results of B.Felzenszwalb [17] to left Goldie near-rings with radical character.

We know that an Artinian ring with unity is a Noetherian ring and a Noetherian ring has no infinite direct sum of ideals and always satisfies the acc on annihilators. On the otherhand, commutative integral domain like $Z[X_i \mid i=1,2,..; X_iX_j = X_jX_i]$ satisfies the later two conditions, yet it is not Noetherian, for we have a strictly ascending chain of ideals viz.,

$$\langle X_1\rangle \subset \langle X_1,X_2\rangle \subset \langle X_1,X_2,X_3\rangle \subset \cdots$$

In this way, a near-ring with acc on annihilators having no infinite direct sum of ideals (near-ring subgroups) need not satisfy the acc or the dcc on its subalgebraic structures, but it may contain some parts satisfying the acc or the dcc on the same.

Here we prove the results ([[37]]) on left Goldie near-rings with parts satisfying the acc or dcc on its subalgebraic structures. The results obtained here may be called an attempt to study what may be termed as the Artinian radical of such a near-ring. One may expect to get more elegent structure theorems in case of a left Goldie near-ring carrying such a non Goldie part in it.

The chapter is devided into four sections. The prerequisites of this chapter are included in the first section. The second section contains some basic results on a near-ring which is radical over a subnear-ring (as defined).

The third one mainly consists of results leading to the existence of classical near-ring of left quotients of a subnear-ring A of a left Goldie near-ring which is radical over A.

The last section contains some interesting results on left Goldie near-rings with parts having minimum conditions. Here we prove a result on a left singular subsets modulo maximal annihilator of a left Goldie near-ring which leads us to the cyclic structure of an ideal I satisfying the dcc on its right N-subgroups. Another important result proved here is a sufficient condition on an ideal (which is minimal as an invariant subnear-ring with dcc on its right N-subgroups) to be a near-ring group over a near-ring with dcc on its near-ring subgroups which is an extension of an epimorphic image of N. Also, if N is a strongly semiprime strictly left Goldie near-ring where every weakly essential left N-subgroup is essential then in case of a countable left ideal I with dcc on its N-subgroups, the N-group $N/_{l(I)}$ also inherits the same character as with I.

4.1. Prerequisites

4.1.1. Definitions. An N-subgroup A of an N-group E is said to be closed if A has no proper essential extension in E. In otherwords, A is a closed N-subgroup of E if for any N-subgroup B of E, $A \subseteq_e B$ implies $A = B$ and we denote it by $A \leq_c E$.

Thus a left N-subgroup A of N is a closed left N-subgroup of N if A has no proper essential extension in N.

4.1.2. Lemma : Let C, D be two N-subgroups of E such that $D \subseteq_e C$ then, for an N-subgroup M ($\subseteq C$) of E, $D \cap M \subseteq_e C \cap M$.

Proof : Let X ($\neq 0$) be an N-subgroup of E such that $X \subseteq C \cap M \subseteq C, M$.

Thus, $M \cap X = X$.

Also, $(D \cap M) \cap X = D \cap (M \cap X)$

$= D \cap X$

$\neq (0)$, (as $D \subseteq_e C$)

Therefore, $D \cap M \subseteq_e C \cap M$. //

4.1.3. Lemma [37] : Let C, D be two N-subgroups of E and B is an ideal of E such that $B \leq_c C$, $B \subseteq D \subseteq_e C$ then $D/_B \subseteq_e C/_B$.

Proof : Let $M/_B$ ($\neq \bar{0}$) be an N-subgroup of $C/_B$ with $M/_B \cap D/_B = (\bar{0})$ (= B)

This gives, $M \cap D = B$.

Now $D \subseteq_e C \Rightarrow M \cap D \subseteq_e M \cap C$ (by 4.1.2)

$\Rightarrow B \subseteq_e M \cap C$

But B, being closed we therefore get $B = M \cap C = M$

Hence $M/_B = (\bar{0})$. Thus $D/_B \subseteq_e C/_B$. //

4.1.4. Lemma : Let P be an ideal of N such that $P \leq_c N$. If a weakly essential left N-subgroup of N is essential then a weakly essential left $\bar{N}$-subgroup of $\bar{N}$ ($= N/_P$) is also essential.

Proof : Let $\bar{A} = A/_P$ be a weakly essential left $\bar{N}$-subgroup of $\bar{N}$. Then by following the steps of 3.1.9(c), we get that A is a weakly essential left N-subgroup of N and by hypothesis, in turn, $A \subseteq_e N$.

Again, since $P \leq_c N$, $P \subseteq A \subseteq N$, by 4.1.3. we get $A/_P \subseteq_e N/_P$.

Thus $\bar{A} \subseteq_e \bar{N}$. //

4.1.5. Note : If d is a non-zero-divisor in N, then

(1) for any $t \in Z^{+}$, d^{t} is also a non-zero-divisor in N.

(2) $\overline{d}$ is also a non-zero-divisor in any factor near-ring $\overline{N}$ of N

4.1.6. Lemma [[14]] : Let N be a strongly semiprime strictly left Goldie near-ring where every weakly essential left N-subgroup is also essential.

Then any subnear-ring A of N satisfies the dcc on left annihilators in A.

Proof : By 3.2.17, N satisfies the dcc on left annihilators. And A, in turn, satisfies the dcc on left annihilators (by 1.4.5.) . //

4.1.7. Definition : An ideal (left/right) I of N is said to be countable if it is countable as a set.

4.1.8. Lemma [[37]] : Let I be a countable left ideal of a strongly semiprime strictly left Goldie near-ring as in 4.1.6. Then there is a finite set S of I such that $l(S) = l(I)$.

Proof : For $y_1 \in I$, $l(\{y_1\}) \supseteq l(I)$

Let $y_2 \in I$, $y_2 \neq y_1$ then

$\{y_1\} \subseteq \{y_1, y_2\} \subseteq I$

$\Rightarrow \quad l(\{y_1\}) \supseteq l(\{y_1,y_2\}) \supseteq l(I)$

Thus we get a descending chain

$l(\{y_1\}) \supseteq l(\{y_1,y_2\}) \supseteq \dots (\supseteq l(I))$ of left annihilators which stops after a finite steps, by 3.2.17.

Suppose $t \in Z^+$ such that

$l(\{y_1,y_2,\dots,y_t\}) = l(\{y_1,y_2,\dots,y_{t+1}\}) = \dots = l(I)$

Thus, if $S = \{ y_1,y_2,\dots,y_t\}$ $(\subseteq I)$ then $l(S) = l(I)$. //

4.2. A-radical near-rings.

If N is a regular near-ring whose idempotents are central then (N +) is abelian [36]. Also we have examples [16] of abelian near-rings whose idempotents are only non nilpotent elements.

In an abelian near-ring N, the subset D of distributive elements of N is a subnear-ring of N (it is actually a ring). Here we note that if in N, non nilpotent elements are distributives then for each $x \in N$, we get a $t \in Z^+$ such that $x^t \in D$. Thus we give the following definition.

4.2.1. Definitions : Let A be a subnear-ring of N. Then N is called A-radical (or N is radical over A) if for each $x \in N$ there exists a $t \in Z^+$ such that $x^t \in A$.

As $1 \in N$, clearly $1 \in A$ if N is A-radical.

If the near-ring N is A-radical then we simply say that N is a radical near-ring.

In the Klein's four group, we note the existence of such a radical near-ring.

4.2.2. Example (E(16), Page 339-[42]): $N = \{0,a,b,c\}$ is a near-ring under addition [defined in the table 1.1(i)] and multiplication defined by the following table.

·	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	0	a

Table : 4.1.

Here $A = \{0,a\}$ is a subnear-ring of N and for each $x \in N$, there exists a $t\ (=2) \in Z^+$ such that $x^t \in A$. Thus N is A-radical.

4.2.3. Lemma [[14]] : Let $k\ (\neq 0) \in N$ be such that for each $x \in N$ there exists $t \in Z^+$ satisfying the condition $kx^tk = 0$. Then there is an $a\ (\neq 0) \in N$ such that $a^2 = 0$ and satisfies the same hypothesis as k.

Proof : If $k^2 = 0$, then we consider $a = k$ and thus k does

If $k^2 \neq 0$, we choose $x = k$. Then by the given condition we get $k^m = 0$ for $m\ (\varepsilon Z^+) \geq 3$.

Now, let α be the minimal of m such that $k^\alpha = 0$.

Then $k^{\alpha-1} \neq 0$.

As $\alpha-1 \geq 2$, it follows that

$2(\alpha-1) = (\alpha-1) + (\alpha-1) \geq (\alpha-1) + 2 = \alpha+1 > \alpha$

Hence $k^{2(\alpha-1)} = 0$. Thus $a = k^{\alpha-1} (\neq 0)$ is such that $a^2 = 0$. Also, for each $x \varepsilon N$, as we have $t \varepsilon Z^+$ with $kx^t k = 0$, we get

$ax^t a = k^{\alpha-1} x^t k^{\alpha-1} = 0$.

Thus $a = k^{\alpha-1} (\neq 0)$ does our job. //

4.2.4. Proposition [14] : Let N be a strongly semiprime near-ring with acc on left annihilators and $k\ (\neq 0)\ \varepsilon N$ is such that for each $x \varepsilon N$ there exists $t \varepsilon Z^+$ with $kx^t k = 0$.

Then there exists an $a\ (\neq 0)\ \varepsilon N$ with $a^2 = 0$ satisfies the same hypothesis as k and $ama = 0$ for an $m\ (\neq 0)\ \varepsilon N$ with $m^2 = 0$.

Proof : By 4.2.3, we have $a\ (\neq 0)\ \varepsilon N$, $a^2 = 0$ such that for each $x \varepsilon N$ there exists a $t \varepsilon Z^+$ with $ax^t a = 0$.

Now, for an $m (\neq 0)\ \varepsilon N$ with $m^2 = 0$, we have $m+maxa (\varepsilon N)$ such that for some $\alpha \varepsilon Z^+$,

$a(m + maxa)^\alpha a = 0$

$$\Rightarrow \quad a\,(m + ma\,xa)^{\alpha-1}\,(m + ma\,xa)\,a = 0$$

$$\Rightarrow \quad a\,(m + maxa)^{\alpha-1}\,ma = 0 \qquad (\text{as } a^2 = 0)$$

$$\Rightarrow \quad a\,(m + maxa)^{\alpha-2}\,(m + maxa)\,ma = 0$$

$$\Rightarrow \quad a\,(m + maxa)^{\alpha-2}\,(maxa)\,ma = 0 \qquad (\text{as } m^2 = 0)$$

$$\Rightarrow \quad a\,(m + maxa)^{\alpha-2}\,(ma)(xama) = 0$$

$$\Rightarrow \quad a\,(m + maxa)^{\alpha-3}\,(ma)(xama)^2 = 0$$

$$\cdots$$

$$\cdots$$

$$\Rightarrow \quad a\,(m + maxa)\,(ma)(xama)^{\alpha-2} = 0 \ \text{[repeating the process]}$$

$$\Rightarrow \quad a\,ma\,(xama)^{\alpha-1} = 0$$

$$\Rightarrow \quad (xama)^{\alpha} = 0$$

Thus Nama is a nil left N-subset of N (for each x ε N, xama (ε Nama) is nilpotent). And by 3.2.7, N cannot have such a non-zero nil left N-subset. Therefore, Nama = (0) which gives (as 1 ε N), ama = 0. //

4.2.5. <u>Proposition</u> [[14]] : Let N be as in 4.2.4. If p,q ε N with pq = 0 then paq = 0 for a ($\neq$ 0) ε N with $a^2 = 0$ and satisfies the same hypothesis as k in 4.2.4.

<u>Proof</u> : By 4.2.3, there exists a ($\neq$ 0) ε N, $a^2 = 0$ such that for each x ε N we have a t ε Z^+ with $ax^ta = 0$

Now, for $y \in N$, $(qyp)^2 = (qyp)(qyp)$

$= (qy)(pq)yp$

$= 0$ (as $pq = 0$)

So, by 4.2.4, we have $a(qyp)a = 0$ where $a (\neq 0) \in N$, $a^2=0$ and a satisfies same hypothesis as k.

Now, $(ypaq)^2 = (ypaq)(ypaq)$

$= (yp)(a(qyp)a)q$

$= 0$ [as $a(qyp)a = 0$]

$\Rightarrow (Npaq)^2 = (0)$

Thus Npaq is a nil left N-subset of N and by 3.2.7, we therefore get Npaq = (0) which gives paq = 0. //

4.2.6. Proposition [[14]] : Let N be a strongly semiprime near-ring with acc on left annihilators. If $k \in N$ such that for each $x \in N$ there exists a $t \in Z^+$ with $kx^t k = 0$ then $k = 0$.

Proof : If possible, let $k \neq 0$.

Then by 4.2.3 we get $a (\neq 0) \in N$ with $a^2 = 0$ such that for each $x \in N$ there exists a $\alpha \in Z^+$ satisfying $ax^\alpha a = 0$.

Thus, $\alpha = 1 \Rightarrow a x a = 0$

$\Rightarrow (xa)^2 = (xa)(xa) = 0$

Suppose, $\alpha > 1$ then $ax^{\alpha}a = 0$ gives

$$(ax^{\alpha-1})(xa) = 0$$

$$\Rightarrow (ax^{\alpha-1})\ a\ (xa) = 0 \qquad \text{[by 4.2.5]}$$

$$\Rightarrow (ax^{\alpha-2})(xa)^2 = 0$$

In like manner, we get $(ax^{\alpha-3})(xa)^3 = 0$. Finally we obtain $(xa)^{\alpha+1} = 0$. And this gives that Na is a nil left N-subset of N. Hence by 3.2.7, $Na = (0)$ giving therbby $a = 0$ (for $1 \in N$) which is not true.

Therefore, $k = 0$. //

4.2.7. Proposition [[14]] : Let N be a strongly semiprime near-ring with acc on left annihilators and A be a subnear-ring of N such that N is A-radical.

If for $a \in A$, $l_A(a) = (0)$ then $l(a) = (0)$.

Proof : Let $x(\neq 0) \in l(a)$

Since N is A-radical, we get a $t \in Z^+$ such that $x^t \in A$.

Now, $x \in l(a)$

$$\Rightarrow xa = 0$$

$$\Rightarrow x^t a = 0$$

$$\Rightarrow x^t \in l_A(a) = (0)$$

$$\Rightarrow x^t = 0 .$$

Thus x is nilpotent. Hence $l(a)$ is a nil left N-subgroup of N. Hence by 3.2.7, $l(a) = (0)$. //

4.2.8. Proposition [[14]] : Let N be as above. Then A cannot have any nonzero nil left A-subset and for $a_1, a_2 \varepsilon A$, $Aa_1 \cap Aa_2 = (0)$ implies $Na_1 \cap Na_2 = (0)$.

Proof : Since N satisfies the acc on left annihilators then by 1.4.5, A also satisfies the acc on left annihilators.

Let B be a nilpotent left A-subset of A such that $B^2 = (0)$. Again for $x \varepsilon N$, we get $t \varepsilon Z^+$ such that $x^t \varepsilon A$ (since N is A-radical).

Therefore, $bx^t b \varepsilon BAB$, for $b \varepsilon B$.

$$\Rightarrow \quad bx^t b \subseteq BB = B^2 = (0)$$

$$\Rightarrow \quad bx^t b = 0.$$

And so by 4.2.6, we get $b = 0$. Hence $B = (0)$. Therefore A has no non-zero nilpotent left A-subset. Similarly we can prove that A has no non-zero nilpotent right A-subset. Thus A is a strongly semiprime near-ring. Moreover, A also satisfies the acc on its left annihilators. Hence by 3.2.7, it has no non-zero nil left N-subset.

Now, let for $a_1, a_2 \varepsilon A$, $Aa_1 \cap Aa_2 = (0)$

and $H = Na_1 \cap Aa_2$.

Consider, $h = x_1a_1 = x_2a_2 \in H$, $x_1 \in N$, $x_2 \in A$, $h \in H$.

Then, for $a_1x_1 \in N$, we have got a $t \in Z^+$ such that $(a_1x_1)^t \in A$.

Now, $(a_1x_1)^t a_1 = \underbrace{(a_1x_1)(a_1x_1)\ \dots\ (a_1x_1)}_{t \text{ times}}\ a_1$

$$= a_1\ (x_1a_1)^t$$

$$= a_1(x_2a_2)^t, \text{ (as } x_1a_1 = x_2a_2)$$

$$\Rightarrow \quad (a_1x_1)^t a_1 = (\ a_1(x_2a_2)^{t-1}x_2)\ a_2 \in Aa_2 \quad \text{(since } a_1, a_2, x_2 \in A)$$

Thus, $(a_1x_1)^t a_1 \in Aa_1 \cap Aa_2 = (0)$

$$\Rightarrow \quad h^{t+1} = (x_1a_1)^{t+1} = \underbrace{(x_1a_1)(x_1a_1)\dots\ (x_1a_1)}_{(t+1) \text{ times}}$$

$$= x_1\ (a_1x_1)^t\ a_1$$

$$= 0$$

So, H is a nil left A-subgroup of A. Hence H = (0) by what we have obtained above. . . . (i)

Therefore, $H = Na_1 \cap Na_2$.

Now, let $y \in Na_1 \cap Na_2$,

Then, $y = y_1a_1 = y_2a_2$, $y_1, y_2 \in N$

By Λ-radical character of N, we get a $t \in Z^+$ such that $(a_2y_2)^t \in \Lambda$.

So, $a_2(y_1a_1)^t = a_2(y_2a_2)^t$

$= (a_2y_2)^t a_2 \in \Lambda a_2$

Also, $a_2(y_1a_1)^t = (a_2(y_1a_1)^{t-1}y_1)\, a_1 \in Na_1$

Therefore, $a_2(y_1a_1)^t \in Na_1 \cap \Lambda a_2 = (0)$ [by (i)]

$\Rightarrow a_2\,(y_1a_1)^t = 0$ . . . (ii)

Thus, $y^{t+1} = (y_1a_1)^{t+1} = (y_1a_1)(y_1a_1)^t$

$= (y_2a_2)(y_1a_1)^t$

$= y_2(a_2(y_1a_1)^t) = 0$ [by (ii)]

So, $Na_1 \cap Na_2$ is a nil left N-subgroup of N . And by 3.2.7, therefore we get $Na_1 \cap Na_2 = (0)$. //

4.2.9. <u>Proposition</u> [[14]] : Let N be as above. If for some $a \in A$, $l_A(a) \neq (0)$ then $Aa \not\subseteq_e A$.

<u>Proof</u> : Suppose $Na \subseteq_e N$. Then, by 3.2.8 and 3.2.11, we get $l(a) = (0)$

$\Rightarrow l_A(a) = l(a) \cap A = (0)$, a contradiction.

So, $Na \not\subseteq_e N$.

If possible, let $Aa \subseteq_e A$ and B is a non-zero left N-subgroup of N. So B can not be nil.

Let $b \ (\neq 0) \in B$ such that b is not nilpotent.

Now, N being A-radical, we get a $t \in Z^+$ such that $b^t \in A$.

As $1 \in A$, $Ab^t \neq (0)$ and therefore

$$Aa \cap Ab^t \neq (0) \quad (\text{as } Aa \subseteq_e A)$$

Again, $Aa \cap Ab^t \subseteq Na \cap Ab^t \subseteq Na \cap B$.

$$\Rightarrow \quad Na \cap B \neq (0)$$

$\Rightarrow \quad Na \subseteq_e N$ which contradicts what we have proved above.

Therefore, $Aa \not\subseteq_e A$. /

4.2.10. <u>Proposition</u> [[14]] : Let N be a strongly prime near-ring with acc on left annihilators. If $a, b \in N$, $b \neq 0$ and for each $x \in N$ there exists a $t \in Z^+$ such that $ax^tb = 0$ then $a = 0$.

<u>Proof</u> : Fix one $x \in N$. Then by hypothesis we get a $t \in Z^+$ such that $ax^tb = 0$.

Consider $B = \{y \in N \mid ax^ty = 0\}$.

So, $b \in B$. Now if $z \in B$ then $ax^tz = 0$

$\Rightarrow \quad (ax^tz)\,n = 0, \quad$ for $n \in N$

$\Rightarrow \quad ax^t(zn) = 0$

$\Rightarrow \quad zn \in B.$

Thus B is a right N-subset of N.

Also, $ax^t z = 0 \Rightarrow z(ax^t z)\ a = 0$

$$\Rightarrow (za)\ x^t (za) = 0$$

Thus za ε N is such that for each x ε N, there exists a $t \in Z^+$ satisfying $(za)\ x^t (za) = 0$.

Moreover N being strongly prime, it is strongly semiprime also. Thus by 4.2.6, we get $za = 0$

Thus, $Ba = (0)$.

Now, $(NB)(NaN) = N(BN)\ aN$

$$\subseteq NB(aN) = N(Ba)N = (0).$$

$$\Rightarrow (NB)(NaN) = (0)$$

$\Rightarrow NB = (0)$ or $NaN = (0)$ [as N is strongly prime]

$\Rightarrow B = (0)$, a contradiction (for $b(\neq 0) \in B$)

Thus, $NaN = (0)$.

Hence $a = 0$ (for $1 \in N$). /

4.2.11. <u>Proposition</u> [[14]] : Let N be strongly prime with acc on left annihilators and A be a subnear-ring of N such that N is A-radical. Then A is strongly prime.

<u>Proof</u> : Let I,J be two invariant subsets of A such that

$IJ = (0)$, $J \neq (0)$.

Since N is A-radical, for each $x \in N$ there exists a $t \in Z^+$ such that $x^t \in A$.

For $a \in I$, $b \in J$, $b \neq 0 \Rightarrow ax^t b \in I\, AJ \subseteq IJ = (0)$

$$\Rightarrow ax^t b = 0$$

So, by 4.2.10, $a = 0$. Therefore $I = (0)$

Hence A is strongly prime. //

4.3. Radical left Goldie near-rings

Throughout the section A will mean a subnear-ring of N such that N is A-radical.

4.3.1. Theorem : In a strongly semiprime strictly left Goldie near-ring N, a ($\in$ A) is a non-zero-divisor in A if and only if a is a non-zero-divisor in N.

Proof : Suppose a ($\in$ A) is a non-zero-divisor in N. Then $l(a) = (0) = r(a)$, which in turn gives $l_A(a) = l(a) \cap A = (0)$ (by 1.3.21) and $r_A(a) = r(a) \cap A = (0)$ (by 1.3.21).

Thus a is a non-zero-divisor in A.

Conversely, if a is a non-zero-divisor in A, then

$$l_A(a) = (0) = r_A(a)$$

$\Rightarrow$ $l(a) = (0)$, (by 4.2.7)

$\Rightarrow$ $Na \subseteq_e N$, (by 3.2.9)

And therefore, by 3.2.14, it follows that a is a non-zero-divisor in N. //

4.3.2. Theorem [[14]] : Let N be strongly prime strictly left Goldie as in 4.1.6. If left annihilators in A are distributively generated, then an essential left A-subgroup of A has a non-zero-divisor.

Proof : By 4.1.6, subnear-ring A satisfies the dcc on left annihilators.

Let B be an essential left A-subgroup of A.

Choose $a \in B$ so that $l_A(a)$ is minimal in A.

Now, $l_A(a) \neq (0)$

$\Rightarrow$ $Aa \not\subseteq_e A$, [by 4.2.9]

So, there is a left A-subgroup J ($\neq 0$) of A such that $Aa \cap J = (0)$.

Again, $B \subseteq_e A \Rightarrow B \cap J \neq (0)$.

Thus, J_1 ($= B \cap J$) is a non-zero left A-subgroup of A.

Also, $Aa \cap J_1 \subseteq Aa \cap (B \cap J) = (0)$ ... (i)

$\Rightarrow$ $Aa \cap J_1 = (0)$

If $x \in J_1$ and $d \in l_A(x+a)$ then $d(x+a) = (0)$.

By hypothesis, $d = \sum_{fin} \pm s_i$, where $s_i \in S$ (the set of distributive elements) and $l_A(x+a) = \langle S \rangle$.

Then $s_i(x+a) = 0$, for each i

$$\Rightarrow s_i x = (-s_i)\, a$$

$$\Rightarrow (\Sigma\, s_i)\, x = (\Sigma -s_i)a \in Aa, \text{ for } s_i \in A.$$

Also, $x \in J_1 \Rightarrow (\Sigma \pm s_i)x \in J_1 \cap Aa = (0)$, [by (i)]

$$\Rightarrow dx = 0$$

$$\Rightarrow d \in l_A(x)$$

Again, $da = (\Sigma \pm s_i)\, a = (\Sigma \pm s_i)\, x$

$$= \Sigma \pm s_i x \in J_1$$

Thus, $da \in J_1 \cap Aa = (0)$

$$\Rightarrow da = 0$$

$$\Rightarrow d \in l_A(a)$$

$$\Rightarrow d \in l_A(a) \cap l_A(x).$$

Thus, $l_A(x+a) \subseteq l_A(a) \cap l_A(x) \subseteq l_A(a)$

As $l_A(a)$ is minimal, it follows that

$l_A(x+a) = l_A(a) = l_A(a) \cap l_A(x) \subseteq l_A(x)$

$\Rightarrow \; l_A(a)x = (0)$

$\Rightarrow \; l_A(a)\; J_1 = (0)$, (for $x \in J_1$)

So, $(l_A(a)A)\;(J_1A) = l_A(a)\;(AJ_1)\;A$

$\subseteq l_A(a)\; J_1A$, (as J_1 is a left A-subgroup of A)

$= (0)$

Now, $J_1A \neq (0)$ and each of $l_A(a)\; A$ and J_1A is an invariant subsets of A. Since A is strongly prime by 4.2.11, it follows that $l_A(a)\; A = (0)$ and hence $l_A(a) = (0)$ (as $1 \in A$).

Thus by using 4.2.7, we get $l(a) = (0)$.

Then applying 3.2.9 and 3.2.14 according we get that a is a non-zero-divisor in N.

Hence by 4.3.1, a is a non-zero-divisor in A. //

4.3.3. Theorem [[14]] : In a strongly prime strictly left Goldie near-ring as in 4.3.2, the subnearring A satisfies the Ore condition w.r.t.S (the set of non-zero-divisors in A).

Proof : Let a,b $\in$ A with a $\in$ S.

Then $l_A(a) = (0) \Rightarrow l(a) = (0)$

$\Rightarrow Na \subseteq_e N$, (by 3.2.9)

Suppose, $Aa \not\subseteq_e A$.

Then we have a left A-subgroup $C\ (\neq 0)$ of A such that $Aa \cap C = (0)$.

Then $Aa \cap Ax \subseteq Aa \cap C = (0)$, (for an $x\ (\neq 0) \in C$)

$\Rightarrow Aa \cap Ax = (0)$

$\Rightarrow Na \cap Nx = (0)$, (by 4.2.8).

$\Rightarrow Nx = (0)$ (as $Na \subseteq_e N$)

$\Rightarrow x = 0$, a contradiction.

Thus, $Aa \subseteq_e A$.

Consider $(Aa \,;\, b) = \{ x \in A \mid xb \in Aa \}$ for $b \in A$.

By 3.1.8, $(Aa \,;\, b) \subseteq_e A$ and so by 4.3.2 it contains an element $a_1 \in S$.

And thus we get $a_1 b = b_1 a$ for some $b_1 \in A$ and hence Ore condition is satisfied w.r.t.S. //

Finally, we come to our main result which follows from Tiwary and Seth [53] as discussed in Chapter III, together with what we have shown above.

4.3.4. <u>Theorem</u> [[14]] : Let N be a strongly prime strictly left Goldie d.g.nr. such that a weakly essential left N-subgroup of N is essential.

If N is A-radical and A has distributively generated left annihilators only, then the complete near-ring of left quotients of A is a classical near-ring of left quotients of A w.r.t. the set of distributive non-zero-divisors. //

4.4. Left Goldie near-rings with its parts having minimum conditions.

4.4.1. Definition : A near-ring N satisfies the left essential descending chain condition (l.e.dcc) if any descending chain of essential left N-subgroups of N stops after a finite steps.

4.4.2. Proposition [[37]] : If N is with l.e.dcc and P is an ideal of N then $N/_P$ is also with l.e.dcc.

Proof : Consider the natural epimorphism

$$f : N \to N/_P \ (= \overline{N}).$$

By 3.1.9(b), if X is an essential left $\overline{N}$-subgroup of $\overline{N}$ then $f^{-1}(X)$ is an essential left N-subgroup of N. Consider a descending chain

$$\overline{I}_1 \supseteq \overline{I}_2 \supseteq \overline{I}_3 \supseteq \cdots \text{ of essential left } \overline{N}\text{-subgroup of } \overline{N}$$

(where $\overline{I}_i = I_i/_P$).

Then $f^{-1}(\overline{I}_1) \supseteq f^{-1}(\overline{I}_2) \supseteq f^{-1}(\overline{I}_3) \supseteq \cdots$

Or, $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ is a descending chain of essential left N-subgroup of N.

And by hypothesis, we get a $t \in Z^+$ such that

$I_t = I_{t+1} = \ldots$ which gives

$\bar{I}_t = \bar{I}_{t+1} = \ldots$ //

4.4.3. <u>Definition</u> : A near-ring N is said to be a <u>quotient near-ring</u> if for a non-zero-divisor d of N, N = Nd.

4.4.4. <u>Theorem</u> [[37]] : If N is a strictly left Goldie near-ring with l.e.dcc then N is a quotient near-ring.

<u>Proof</u> : Let d be a non-zero-divisor in N. Then each d^i, $i \in Z^+$ is also a non-zero-divisor.

So, by 3.2.9, $Nd^i \subseteq_e N$, for each i.

Also, $Nd \supseteq Nd^2 \supseteq Nd^3 \supseteq \ldots$

Since N is with l.e.dcc, we have a $t \in Z^+$ such that

$Nd^t = Nd^{t+1} = \ldots$

$\Rightarrow (N - Nd)\, d^t = (0)$

$\Rightarrow N - Nd = (0)$ (as d^t is a non-zero-divisor)

$\Rightarrow N = Nd$

Thus N is a quotient near-ring. //

Now 3.4.2, 4.4.2 together with 4.4.4 give us the following

4.4.5. Theorem [[37]] : If N is a strongly semiprime strictly left Goldie near-ring with l.e.dcc and $P \in \Gamma$ then $N/_P$ is a quotient near-ring. //

In what follows N will be a strongly semiprime strictly left Goldie d.g.nr. with distributively generated left annihilators and a weakly essential left N-subgroup of N is essential [as in 3.5].

4.4.6. Theorem [[37]] : If I ($\neq 0$) is an ideal which is minimal as an invariant subnear-ring of N such that $P(=l(I) \in \Gamma)$ is closed then $Z(I)$ [= $\{x \in I \mid cx = 0$, for some non-zero-divisor $\bar{c}$ in $\bar{N}$ (= $N/_P$)] = (0).

Proof : We first claim that P is strongly prime.

For this, let A,B be two invariant subsets of N with $AB \subseteq P$ and suppose $B \not\subseteq P$.

Then $ABI = (0)$, but $BI \neq (0)$.

$\Rightarrow \quad A \subseteq l(BI)$

Since $BI \subseteq I$ (as I is an ideal) and BI is an invariant subnear-ring of N, we get

$BI = I$ (for I minimal)

$\Rightarrow \quad l(BI) = l(I) = P$

$\Rightarrow \quad A \subseteq P$, giving thereby P strongly prime.

Now, as in the proof of 3.4.2, it follows that $N/_P$ is a strongly prime strictly left Goldi near-ring.

If A is an essential left N-subgroup of N with $P \subseteq A \subseteq N$ and since $P = l(I)$ is closed, it follows from 4.1.3. that $A/P \subseteq_e N/P$.

Now let $x \in Z(I)$. Then $cx = 0$ for some non-zero-divisor $\bar{c}(= c+P)$ in $\bar{N}$ $(= N/P)$.

Again by 3.4.2, $\bar{N}$ is a strongly prime strictly left Goldie near-ring.

This gives, $\bar{N}\bar{c} \subseteq_e \bar{N}$ (by 3.2.9)

And therefore, $(\bar{N}\bar{c} \,;\, \bar{n}) \subseteq_e \bar{N}$ for each $\bar{n} \in \bar{N}$ (by 3.1.8).

Now, if $l(\bar{T})$ is a left annihilator in $\bar{N}$ then by 3.3.5, $l(\bar{T}) = l(H)/P$, where $H \subseteq N$.

Also, since $l(H) = \langle S \rangle$ for a set S of distributive elements of N, clearly we get

$$l(\bar{T}) = \langle \{s+P \mid s \in S\} \rangle$$

and $\{s + P \mid s \in S\}$ is again a set of distributive elements of $\bar{N}$. Thus a left annihilator in $\bar{N}$ is also distributively generated.

Again, by 4.1.4, a weakly essential left $\bar{N}$-subgroup of $\bar{N}$ is also essential.

Hence $\bar{N}$ satisfies all the conditions of 3.4.5.

Now, if K is an essential left N-subgroup of N with $P \subseteq K \subseteq N$ and since P is closed, it follows from 4.1.3 that $\bar{K} \subseteq_e \bar{N}$.

Thus by 3.4.5, $\bar{K}$ contains a non-zero-divisor $\bar{c}$ (say).

Therefore, $cx = 0$ for $x \in Z(I)$.

Again, since $\bar{c}$ is a non-zero-divisor in $\bar{N}$ so, by 3.2.9,

$\bar{N}\bar{c} \subseteq_e \bar{N}$.

And therefore, $(\bar{N}\bar{c} \,;\, \bar{n}) \subseteq_e \bar{N}$ for each $\bar{n} \in \bar{N}$ (by 3.1.8.)

Also, by 3.4.5, $(\bar{N}\bar{c} \,;\, \bar{n})$ contains a non-zero-divisor $\bar{d}$(say).

Thus, $\bar{d}\,\bar{n} \in \bar{N}\,\bar{c}$

$\Rightarrow \overline{dn} = \overline{uc}$ for some $\bar{u} \in \bar{N}$

$\Rightarrow dn - uc \in P = l(I)$

$\Rightarrow (dn - uc)\, I = (0)$

$\Rightarrow (dn - uc)\, x = 0$ for $x \in Z(I) \subseteq I$

$\Rightarrow (dn)\, x - (uc)\, x = 0$

$\Rightarrow d(nx) - u\,(cx) = 0$

$\Rightarrow d(nx) = 0$ (as $cx = 0$, by our choice).

$\Rightarrow nx \in Z(I)$, (as $\bar{d}$ is a non-zero-divisor in $\bar{N}$)

So, $Z(I)$ is a left N-subset of N.

Again, if $x \in Z(I)$ then we have $cx = 0$, for some non-zero-divisor $\bar{c} \in \bar{N}$.

This gives, $\bar{c}\,\bar{x} = \bar{0}$

$\Rightarrow \bar{x} = \bar{0}$ (as $\bar{c}$ is a non-zero-divisor)

$\Rightarrow x \in P$

$\Rightarrow \quad Z(I) \subseteq P = l(I)$

$\Rightarrow \quad (Z(I))\, I = (0)$

Now, $(Z(I))^2 = (Z(I))\,(Z(I))$

$\subseteq (Z(I))\, I = (0)$

$\Rightarrow \quad (Z(I))^2 = (0)$

But N being strongly semiprime, by 3.2.5, it follows that $Z(I) = (0)$. //

4.4.7. <u>Theorem</u> [[37]] : Let I be an ideal of N as above with dcc on its right N-subgroups, then for a non-zero-divisor $\bar{d} \in \bar{N}$ ($P = l(I)$ closed) with d distributive, $I = dI$.

<u>Proof</u> : Since $\bar{d} \in \bar{N}$ is a non-zero-divisor such that d is distributive and so each $d^i I$ is a right N-subgroup of $I(i \in Z^+)$.

Again, $I \supseteq dI \supseteq d^2 I \supseteq \ldots$ gives a $t \in Z^+$ such that $d^t I = d^{t+1} I$ (as I-satisfies dcc on right N-subgroups)

So, for each $x \in I$, we get $y \in I$ such that

$$d^t x = d^{t+1} y$$

$\Rightarrow \quad d^t(x-dy) = 0$

$\Rightarrow \quad x-dy \in Z(I)$, (as $\overline{d^t} = \bar{d}^t$ is a non-zero-divisor)

$\Rightarrow \quad x = dy$, (for $Z(I) = (0)$, by 4.4.6)

$\Rightarrow \quad I \subseteq dI$

And I being an ideal, clearly $dI \subseteq I$.

Thus, $I = dI$. //

4.4.8. Theorem [[37]] : Let N and I be as in 4.4.7. Then I is a near-ring group over a near-ring Q of left quotients of $\overline{N}$($= N/_P$) where Q satisfies the dcc on its near-ring subgroups and is an extension of epimorphic image of N.

Proof : By 3.4.2, $\overline{N}$ is a strongly prime strictly left Goldie near-ring and it is d.g. as N is d.g.

Thus by 3.5.20, $\overline{N}$ has a (classical) near-ring Q (say) of left quotients of $\overline{N}$. Also, by 3.5.22, Q satisfies the dcc on its left Q-subgroups.

Now, if $\bar{d}(\in \overline{N})$ is a non-zero-divisor with d distributive then by 4.4.7, $I = dI$.

$\Rightarrow$ $\bar{I} = \bar{d}\,\bar{I}$

$\Rightarrow$ $(\bar{d})^{-1}\bar{I} = \bar{I}$

Consider the map : $Q \times \bar{I} \to \bar{I}$ defined by $(\bar{d}^{-1}\bar{r}, \bar{x}) \to \bar{d}^{-1}\cdot\overline{(rx)}$,

where $\bar{I} = \{ i + P \mid i \in I\}$

Then this map makes $\bar{I}$ a Q-group.

We again consider the map

$Q \times I \to I$ such that

$(q, i) \to x$ where $q\mathring{i} = x$ such that $\bar{x} = q\bar{i}$.

We first verify that the map is well defined. In otherwords, we get a unique x.

Suppose $x, y \in I$ with $\overline{x} = \overline{y}$ ($= q\,\overline{i}$)

Then $x - y \in P = l(I)$

$\Rightarrow$ $x - y \in l(I) \cap I = (0)$

$\Rightarrow$ $x - y = 0$

$\Rightarrow$ $x = y$.

Now, let $q_1, q_2 \in Q$, $i \in I$ and suppose, $q_1\overline{i} = \overline{x_1}$ and $q_2\,\overline{i} = \overline{x_2}$ (where $q_1 i = x_1$, $q_2 i = x_2$).

As $\overline{I}$ is a Q-group, we get

$(q_1 + q_2)\,\overline{i} = q_1\overline{i} + q_2\overline{i} = \overline{x_1} + \overline{x_2} = \overline{x_1 + x_2}$

So, by definition, $(q_1 + q_2)\, i = x_1 + x_2 = q_1 i + q_2 i$

Again, $(q_1 q_2)\,\overline{i} = q_1\,(q_2\overline{i})$, (as $\overline{I}$ is a Q-group).

Suppose, $(q_1 q_2)\,\overline{i} = \overline{x_3}$ then $(q_1 q_2)\, i = x_3$.

Now, $\overline{x_3} = (q_1 q_2)\,\overline{i} = q_1(q_2\overline{i}) = q_1\overline{x_2}$

$\Rightarrow x_3 = q_1 x_2$

$\Rightarrow (q_1 q_2)\, i = q_1(q_2 i)$

Obviously, for 1(unity) $\in Q$, $1.i = i$, ($i \in I$).

Thus I is a Q-group where Q is the (classical) near-ring of left quotients of $\overline{N}$ which is an epimorphic image of N. //

4.4.9. <u>Theorem</u> [[37]] : Let N be a strongly semiprime strictly left Goldie near-ring where every weakly essential left N-subgroup is essential .

If the countable ideal I satisfies the dcc on N-subgroups of ${}_N I$ then N-group $N/l(I)$ also satisfies the dcc on its N-subgroups.

<u>Proof</u> : I being countable, by 4.1.8, there is a finite set $S = \{y_1, y_2, \ldots, y_t\} \subseteq I$ such that

$$l(I) = l(S) \text{ , } (l(S) \text{ minimal})$$

Now, $y_i \in S \Rightarrow l(y_i) \supseteq l(S)$ for each i.

$$\Rightarrow l(y_1) \cap l(y_2) \cap \ldots \cap l(y_t) \supseteq l(S).$$

And l(S) being minimal, if follows that

$$l(S) = l(y_1) \cap l(y_2) \cap \ldots \cap l(y_t)$$

Let us consider the map

$$f : N \to Ny_1 \oplus Ny_2 \oplus \ldots \oplus Ny_t \text{ such that}$$

$$f(n) = (ny_1, ny_2, \ldots, ny_t), \text{ for } n \in N.$$

Clearly, for any a,b ε N, we have

$$f(a+b) = f(a) + f(b) \text{ and}$$

$$f(na) = n\, f(a), \quad \text{for } n \in N.$$

Thus, f is an N-homomorphism.

Moreover, $x \in \text{Ker } f \Leftrightarrow (xy_1, xy_2, \ldots, xy_t) = (0,0,\ldots,0)$

$\Leftrightarrow xy_i = 0, \quad (\text{for } i = 1,2,\ldots, t)$

$\Leftrightarrow x \in l(y_i)$, for each i

$\Leftrightarrow x \in l(y_1) \cap \ldots \cap l(y_t) = l(S)$

Therefore, $\text{Ker } f = l(S) = l(I)$

So, $f(N) \cong N/_{l(I)}$

Again, we define $\phi : N/_{l(I)} \to Ny_1 \oplus \ldots \oplus Ny_t$.

such that $\phi(\overline{x}) = (xy_1, xy_2, \ldots, xy_t)$, $(\overline{x} = x + l(I))$

Clearly for any $\overline{a}, \overline{b} \in N/_{l(I)}$ we get

$\phi(\overline{a}+\overline{b}) = \phi(\overline{a}) + \phi(\overline{b})$ and

$\phi(n\overline{a}) = n\,\phi(\overline{a})$, $(n \in N)$

And so ϕ is an N-homomorphism.

Again let $\overline{a}, \overline{b} \in N/_{l(I)}$, $\phi(\overline{a}) = \phi(\overline{b})$

Then, $(ay_1, ay_2, \ldots, ay_t) = (by_1, by_2, \ldots, by_t)$

$\Rightarrow ay_i = by_i, \quad (\text{for } i = 1,2,\ldots, t)$

$\Rightarrow (a-b)\, y_i = 0$

$\Rightarrow a-b \in l(y_i), \quad (\text{for } i = 1,2,\ldots, t)$

$\Rightarrow a-b \in l(y_1) \cap l(y_2) \cap \ldots \cap l(y_t) = l(I)$

$\Rightarrow \overline{a} = \overline{b}$.

Hence $\emptyset$ is a monomorphism. So $N/_{1(I)}$ can be embedded in $Ny_1 \oplus Ny_2 \oplus \cdots \oplus Ny_t$ as an N-group.

Now, $y_i \in I \Rightarrow Ny_i \subseteq I$. Thus each Ny_i is an N-subgroup of I and every N-subgroup of Ny_i is also an N-subgroup of I. As I satisfies the dcc on its N-subgroups, each Ny_i also satisfies the dcc on its N-subgroups. Thus the direct sum $Ny_1 \oplus \cdots \oplus Ny_t$ inherits the same character.

Since $\emptyset$ is an embedding, $N/_{1(I)}$ can be considered as an N-subgroup of $Ny_1 \oplus Ny_2 \oplus \cdots \oplus Ny_t$.

Therefore $N/_{1(I)}$ satisfies the dcc on its N-subgroups. //

CHAPTER V

Near-rings with acc on right annihilators

In this chapter we confine ourselves to near-rings with acc on right annihilators; in particular what is termed as a weakly right Goldie near-ring [52] . With the idea of a weakly regular near-ring we prove some interesting structure theorems in case of such a near-ring in the light of A. Oswald [41].

Moreover above mentioned weakly right Goldie structure of a near-ring gives rise to a factor near-ring which is a quasi near-domain.

This chapter has four sections of which the first one contains the prerequisites of the chapter and the second one contains the properties of weakly regular near-rings.

The properties of near-rings with acc on right annihilators are discussed in the third section. The structure theorem viz., a weakly regular d.g.nr. with acc on right annihilators is a direct sum of ideals which are weakly regular simple d.g.nr. with identities is proved in this section.

The fourth section contains the results on strongly prime near-rings with acc on right annihilators. And finally we prove the following result.

In case of a strongly prime weakly right Goldie d.g.nr. with distributively generated left annihilators, if J is a maximal right annihilator and M is the left annihilator of J then in some special cases, $M/(J \cap M)$ is a quasi near-domain.

5.1. Prerequisites :

5.1.1. Lemma : If N is a strongly prime near-ring then $l(N) = (0) = r(N)$.

Proof : We have, $l(N) \subseteq N$.

$\Rightarrow \quad (l(N))^2 \subseteq l(N)N = (0)$

$\Rightarrow \quad (l(N))^2 = (0)$

$\Rightarrow \quad l(N) = (0)$, as $l(N)$ is an invariant subset and N is strongly prime.

Similarly, $r(N) = (0)$. //

5.1.2. Lemma : Let N be strongly prime near-ring and $I\ (\neq 0)$ be a right N-subset of N and $J(\neq 0)$ be a left N-subset of N.

Then $r(I) = (0) = l(J)$

Proof : I being a right N-subset of N, we get $IN \subseteq I$

Again, $Ir(I) = (0)$

$\Rightarrow \quad (NI)\ r(I) = (0)$

$\Rightarrow$ $NI = (0)$ or $r(I) = (0)$, as N strongly prime.

$\Rightarrow$ $I \subseteq r(N) = (0)$ or $r(I) = (0)$

$\Rightarrow$ $I = (0)$ or $r(I) = (0)$

By supposition, $I \neq (0)$, so $r(I) = (0)$.

Similar arguments give that $l(J) = (0)$. //

5.1.3. __Lemma__ : In a prime near-ring N, for any right N-subset I ($\neq$ 0) and left N-subset J ($\neq$ 0) of N, $I \cap J \neq (0)$.

__Proof__ : Suppose $I \cap J = (0)$

Then $IJ \subseteq I, J$.

$\Rightarrow$ $IJ \subseteq I \cap J = (0)$

$\Rightarrow$ $I \subseteq l(J) = (0)$, (by 5.1.2)

$\Rightarrow$ $I = (0)$, not true

Hence $I \cap J \neq (0)$. //

Now, we note the following :

5.1.4. __Note__ : If I, J are normal subgroups of (N,+) such that $I \oplus J = N$ then $i + j = j + i$ for $i \in I$, $j \in J$.

5.1.5. __Lemma__ : Let I,J be normal subgroups of (N,+) such that $I \oplus J = N$. If B is a normal subgroup of I (or J) then B is a normal subgroup of (N, +).

Proof : Let B be a normal subgroup of I.

Now, by 5.1.4, $i + j = j + i$ for $i \in I$, $j \in J$.

Consider $n = i + j$, for $n \in N$.

If $b \in B$ then

$$-n + b + n = -(i+j) + b + (i+j)$$

$$= -j-i+b+i+j$$

$$= -i+b+i \in B$$ as B is a normal subgroup of I.

Thus B is a normal subgroup of N. //

5.1.6. Lemma (Pilz [42], 2.6(b)) : If $\sum_{k \in Z^+} I_k$ is the direct sum of ideals of a near-ring N then elements of different I_k's have product 0.

5.1.7. Lemma : Let I, J be two ideals of N such that $I \oplus J = N$ and I has an identity e. If B is an ideal of I then B is also an ideal of N.

Proof : By 5.1.5, B is a normal subgroup of N.

Let $b \in B \subseteq I$ then $bn \in I$ for $n \in N$.

Thus, $bn = (bn)e = b(ne)$

$= b((i+j)e)$, for $n = i+j$, $i \in I$, $j \in J$.

$= b\,(ie + je)$

$= b(ie)$, (by 5.1.6, $je = 0$)

$\Rightarrow bn = bi \in B$.

Again, for $n, n_1 \in N$, we have

$n(n_1+b) - nn_1 \in I$, as I is an ideal of N.

Thus, $n(n_1+b) - nn_1$

$= [n(n_1+b) - nn_1]e$

$= n(i_1+j_1+b)e - n(i_1+j_1)e$, where $n_1 = i_1+j_1$ for $i_1 \in I, j_1 \in J$

$= n(i_1+b) - ni_1$, (as $j_1e = 0$ by 5.1.6)

$= (i+j)(i_1+b) - (i+j)i_1$, where $n = i+j$

$= i(i_1+b) + j(i_1+b) - (ii_1+ji_1)$

$= i(i_1+b) - ii_1$, as $ji_1 = 0$, $j(i_1+b) = 0$ by 5.1.6.

$\Rightarrow n(n_1+b) - nn_1 = i(i_1+b) - ii_1 \in B$, as B is an ideal of I.

Hence B is an ideal of N. //

5.2. Weakly regular near-rings

First we prove the following lemma.

5.2.1. Lemma : Let N be a regular near-ring. If A is a right (left) N-subset of N, then $A^2 = A$.

Proof : Let $a \in A \subseteq N$ then there exists an element $x \in N$ such that

$a = axa = (ax)\ a$

$= ba$, where $b = ax \in A$

$\Rightarrow \quad a = ba \in A.A = A^2$

$\Rightarrow \quad A \subseteq A^2$

Also, for $a,b \in A$, $ab \in A^2$. But $ab \in A$.

Hence $A^2 \subseteq A$ which gives $A^2 = A$. //

5.2.2. Example (G(19), Page 340-341[42]) :

$N = \{0,1,2,3,4,5\}$ is a near-ring under addition modulo 6 and multiplication defined by the following table.

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	3	4	3	0	1
2	0	0	2	0	0	2
3	0	3	0	3	0	3
4	0	0	4	0	0	4
5	0	3	2	3	0	5

Table 5.1.

Here $\{0,3\} = A$, $\{0,2,4\} = B$ and N are invariant subnear-rings of N. Also, $A^2 = A$, $B^2 = B$, $N^2 = N$. But it contains a non regular element 1. Moreover, each of A,B,N is ideal also.

In the above example, we note that for each invariant subnear-ring J of N, $J^2 = J$. But there is an element $x \in N$ such that x is not regular. So N is not a regular near-ring.

Now, we define the following.

5.2.3. Definition : A near-ring N is called weakly regular if for any ideal I of N, each left I-subgroup A of I, $A^2 = A$.

So, the near-ring in the example 5.2.2, is weakly regular.

Moreover, if N is regular then clearly it is weakly regular.

5.2.4. Lemma : Let I and J be two ideals of a weakly regular near-ring N such that $N = I \oplus J$ and identity $e \in I$ then I is also a weakly regular near-ring (as a subnear-ring of N).

Proof : I being an ideal of N, it is also a near-ring.

Let I_1 be an ideal of I and A be any left I_1-subgroup of I_1. As $N = I \oplus J$, by 5.1.7, we get that I_1 is an ideal of N also. So $A^2 = A$, where A is any left I_1-subgroup of I_1.

Hence I is weakly regular. //

5.2.5. Lemma : A weakly regular near-ring N is a strongly semiprime near-ring.

Proof : Let A be any invariant subset of N such that $A^2 = (0)$.

Now, let $a \in A \subseteq N$ then Na is a left N-subgroup of N (by 1.2.17).

But $(Na)^2 = (Na)(Na) \subseteq A.A$ as $Na \subseteq A$.

$\Rightarrow \quad (Na)^2 \subseteq A^2 = (0)$

$\Rightarrow \quad (Na)^2 = (0)$

Again, Na being a left N-subgroup of the ideal N then we have $(Na)^2 = Na$ (as N weakly regular)

Thus $Na = (Na)^2 = (0)$

$\Rightarrow \quad a = 0$ (as $1 \in N$)

$\Rightarrow \quad A = (0)$

Hence N has no non-zero nilpotent invariant subset.

Therefore, N is strongly semiprime. //

5.2.6. Lemma : Every invariant subset of a regular near-ring N is regular.

Proof : Let A be an invariant subset of a regular near-ring N and $a \in A$ ($\subseteq N$).

N being regular, we have $x \in N$ such that $a = axa$.

Now, $xax = x(ax)$

$= xb$, where $b = ax \in A$.

Write $c = xax = xb \in A$

Then $aca = a(xax)\, a$

$= (axa)\, xa$

$= axa$

$= a$

Thus a is a regular element of A. So A is regular. //

5.2.7. <u>Lemma</u> : A regular near-ring N is weakly regular.

<u>Proof</u> : Let I be an ideal of a regular near-ring N and A be a left I-subgroup of I.

By 5.2.6, I is regular (I being an invariant subset of N).

Now let $a \in A\ (\subseteq I)$. So we have an element $x \in I$ such that $axa = a$.

$\Rightarrow\ a = a(xa) = ab,\ (b = xa \in A)$

$\Rightarrow\ a = ab \in A^2$

$\Rightarrow\ A \subseteq A^2$.

Also, $A^2 \subseteq A$ giving thereby $A^2 = A$.

Hence N is weakly regular. //

5.3. Weakly regular near-rings with acc on right annihilators.

5.3.1. Proposition : Let N be a d.g.nr. satisfying the acc on right annihilators in N. Then any left ideal I of N has a left identity if and only if $x \in Ix$ for every $x \in I$.

Proof : Assume that e ($\in I$) is a left identity of I. Then for each $x \in I$, we have

$$x = ex \in Ix .$$

Conversely, assume that for each $x \in I$, $x \in Ix$. Let $N = \langle S \rangle$, S is a set of distributive elements of N.

For $a \in I$, write $R(a) = \{s - sa \mid s \in S\}$.

Since N is with acc on right annihilators and I, being a left ideal, is a subnear-ring of N. By 1.4.5, I is also with acc on right annihilators in I.

So in the family $\{r_I(R(a)) \mid a \in I\}$ of right annihilators, we can choose a maximal one (say) $r_I(R(e))$ for $e \in I$.

Clearly, $r_I(R(e)) = \{x \in I \mid R(e)x = (0)\} \subseteq I$.

We claim, $I = r_I(R(e))$.

If possible, let $y \in I$ with $y \notin r_I(R(e))$.

Then $R(e).\ y \neq (0)$

$\Rightarrow (s-se)y \neq 0$ for some $s \in S$.

$\Rightarrow \quad s(y-ey) \neq 0 \quad$ (s distributive)

$\Rightarrow \quad y - ey \neq 0 \quad$ (otherwise $s(y-ey) = 0$)

But $y - ey \in I \quad$ (as $y \in I$, a left ideal)

$\Rightarrow \quad y - ey \in I\,(y - ey) \quad$ (by hypothesis)

$\Rightarrow \quad y - ey = e_1(y - ey), \quad$ for some $e_1 \in I$

$\Rightarrow \quad y = e_1(y - ey) + ey$

$= (\Sigma \pm s_i)(y-ey) + ey$, where $e_1 = \Sigma \pm s_i$, $s_i \in S$.

$= \Sigma \pm (s_i - s_i e)\; y + ey$

$= [\; \Sigma \pm (s_i - s_i e) + e \;]\; y$

$= fy,$

where $f = \Sigma \pm (s_i - s_i e) + e$

$= \Sigma \pm (s_i + u_i) + e, \quad u_i = - s_i e \in I$

$= \Sigma \pm s_i + u + e$, for some $u \in I$ (I being a normal subgroup of $(N,+)$)

$\Rightarrow \quad f = e_1 + u + e \in I$

Thus $y = fy$ with $f \in I$.

Now, for any $s - sf \in R(f)$, $f \in I$, $s \in S$.

$$(s - sf)\; y = sy - sfy$$
$$= sy - sy$$
$$= 0$$

$\Rightarrow \quad y \in r_I\ (R(f))$

Thus, $y \in r_I\ (R(f))$ but $y \notin r_I(R(e))$.

On the otherhand, $r_I(R(e)) \subseteq r_I(R(f))$.

For, if $z \in r_I(R(e))$ then $R(e)\ z = (0)$

$\Rightarrow \quad (s - se)\ z = 0, \quad s \in S$

$\Rightarrow \quad fz = [\ \Sigma \pm (s_i - s_i e) + e]\ z$

$= \Sigma \pm (s_i - s_i e)\ z + ez$

$= ez$ (since $(s_i - s_i e)\ z = 0$ for each i)

$\Rightarrow \quad fz = ez$

Now, $(s - sf)\ z = sz - sfz$

$= sz - s(fz)$

$= sz - s(ez)$

$= (s - se)\ z$

$= 0$ (by above)

$\Rightarrow z \in r_I(R(f))$.

Thus $r_I(R(e)) \subseteq r_I\ (R(f))$.

Since $y \in r_I\ (R(f))$ with $y \notin r_I(R(e))$, we get

$r_I(R(e)) \subset r_I(R(f))$ and this contradicts the maximality of $r_I(R(e))$. Thus $I = r_I(R(e))$.

We now prove that e (ε I) is a left identity of I.

For this, let $m \varepsilon I$ then $R(e)\, m = (0)$

$\Rightarrow \quad (s - se)\, m = 0$, for $s \varepsilon S$

$\Rightarrow \quad sm - sem = 0$

$\Rightarrow \quad s(m-em) = 0$, (s is distributive)

Now, for any $n \varepsilon N$, $n = \Sigma \pm s_i$, $s_i \varepsilon S$.

$$\Rightarrow \quad n\,(m - em) = (\Sigma \pm s_i)(m - em)$$

$$= \Sigma \pm s_i(m - em)$$

$$= 0, \quad (\text{as } s_i(m - em) = 0 \text{ for each } i)$$

$\Rightarrow \quad N(m-em) = (0)$

$\Rightarrow \quad I\,(m - em) = (0)$, ($I \subseteq N$)

But $m - em \varepsilon I$. So by assumption,

$$m - em \varepsilon I\,(m - em) = (0)$$

$\Rightarrow \quad m = em$.

Therefore e is a left identity of I. //

Considering $I = N$ above, we get easily the following

5.3.2. <u>Corollary</u> : A d.g.nr. N with acc on right annihilators has a left identity if and only if $x \varepsilon Nx$ for each $x \varepsilon N$.

5.3.3. <u>Proposition</u> : Let N be a weakly regular d.g.nr. with acc on right annihilators then every ideal I of N possesses an identity which is central idempotent of N.

<u>Proof</u> : Let A be a left I-subgroup of I and $x \in A$ ($\subseteq I$).

Then $I(Nx) \subseteq (IN)x \subseteq Nx \subseteq I$, (I being ideal).

Also, we have by 1.2.17, Nx is a left N-subgroup of N. Hence Nx is a left I-subgroup of I as $I(Nx) \subseteq Nx$.

Since N is weakly regular and Nx is a left I-subgroup of I then $(Nx)^2 = Nx$ (by definition).

Thus $Nx = (Nx)^2 = (Nx)(Nx) = (NxN)x$

But $NxN \subseteq I$ as I is an ideal of N and $x \in I$.

Hence $Nx \subseteq Ix$.

$\Rightarrow \quad Nx \subseteq Ix \subseteq A \subseteq I$

$\Rightarrow \quad NA \subseteq A \subseteq I$

Therefore, A is a left N-subgroup of N contained in I. So each left I-subgroup of I is a left N-subgroup of N contained in I. Hence, each left I-subgroup of I generated by $x (\in I)$ is also a left N-subgroup of N and it is contained in I. Symbolically,

$$I^{\langle x\rangle} \subseteq N^{\langle x\rangle} \subseteq I$$

On the otherhand, it is obvious that

$$N^{\langle x\rangle} \subseteq I^{\langle x\rangle}$$

Thus, $_N\langle x\rangle = {}_I\langle x\rangle \subseteq I$.

Now, $x \in {}_I\langle x\rangle = {}_N\langle x\rangle = {}_N\langle x\rangle \; {}_N\langle x\rangle$ (as $_N\langle x\rangle$ is a left I-subgroup of I and N is weakly regular).

Therefore, we suppose that $x = uv$ where $u,v \in {}_N\langle x\rangle$.

Also, let $u = \Sigma \pm s_i$ and $v = \Sigma\, r_j x$ where $s_i \in S$, a generating set of distributive elements and $r_j \in N$.

$$\text{Hence } x = uv = (\Sigma \pm s_i)(\Sigma\, r_j\, x)$$

$$= (\Sigma \pm s_i)((\Sigma\, r_j)x)$$

$$= ((\Sigma \pm s_i)(\Sigma\, r_j))\, x \in Nx$$

$$\Rightarrow x \in Nx \subseteq Ix$$

N being d.g.nr. with acc on right annihilators and $x \in Ix$ for all $x \in I$ then by 5.3.1, I has a left identity e(say).

Now, let $y \in N$ such that $z = ey - ye$

$\Rightarrow\ z \in I$, (I is an ideal of N)

$$\Rightarrow\ ze = (ey - ye)\, e$$

$$= (ey)e - (ye)\, e$$

$$= ye - ye, \text{ (as } e.e = e)$$

$$= 0$$

Thus $I\,(ze)\,I = (0)$.

But $I\,(ze)\,I = (Iz)(eI) = IzI$ (e is left identity of I)

$\Rightarrow \quad IzI = (0)$

Since Iz is a left I-subgroup of I and N is weakly regular, so

$$Iz = (Iz)^2 = (IzI)z = (0)$$

$\Rightarrow \quad ez = 0,\ (e \in I)$

$\Rightarrow \quad z = 0$, (e is left identity of I)

$\Rightarrow \quad ey - ye = 0$

$\Rightarrow \quad ey = ye$

Also, $e^2 = ee = e$. Hence e is central idempotent of N.

Now, if $i \in I$ then

$$ie = ei = i$$

Therefore, e is the identity of I which is central idempotent.//

5.3.4. <u>Corollary</u> : A weakly regular d.g.nr. N with acc on right annihilators in N possesses two sided identity.

<u>Proof</u> : Considering I = N, we immediately get the result from 5.3.3. //

5.3.5. <u>Theorem</u> : A weakly regular d.g.nr. with acc on right annihilators is a direct sum of ideals which are weakly regular simple d.g.nrs. with identities.

<u>Proof</u> : Let I ($\neq 0$) be an ideal of a weakly regular d.g.nr. N with acc on right annihilators then by 5.3.3, I possesses an identity e (say) which is central idempotent of N.

Since $e \in I$, $l_I(e) = (0)$.

$\Rightarrow$ $l(e) \cap I = (0)$ ... (i)

Now, for $n \in N$, $n = (n - ne) + ne$.

$\Rightarrow n \in l(e) + I$, as $ne \in I$ and $(n - ne)e = 0$.

$\Rightarrow N \subseteq l(e) + I$

$\Rightarrow N = l(e) + I$, (as $l(e) + I \subseteq N$)

$\Rightarrow N = l(e) \oplus I$, ... (ii)

Here I is an ideal with identity e and l(e) is also an ideal of N (by 1.3.4 and since $(l(e)\,N)e = l(e)(Ne) = l(e)(eN) = (l(e)\,e)N = (0)$).

Hence by 5.3.3, ideal l(e) also possesses an identity f(say) which is central idempotent of N.

Thus, $f \in l(e) \Rightarrow fe = 0$

$\Rightarrow ef = 0$

$\Rightarrow e \in l(f)$.

Now, let $x \in l(e) \cap l(f)$

$\Rightarrow x \in l(e)$ and $x \in l(f)$

$\Rightarrow$ $x \in l(e)$ and $xf = 0$

$\Rightarrow$ $x = 0$, (f is an identity of $l(e)$).

$\Rightarrow$ $l(e) \cap l(f) = (0)$.

Also let $y\ (\neq 0) \in l(f)$ but $y \notin I$.

Then $y = y + 0$ gives $y \in l(e)$ as $0 \in I$ and $N = l(e) \oplus I$.

Thus, $y \in l(e) \cap l(f) = (0)$

$\Rightarrow$ $y = 0$, a contradiction.

Hence $l(f) \subseteq I$.

$\Rightarrow$ $I = l(f)$, as $I = Ie \subseteq Nl(f) \subseteq l(f)$.

Therefore, every ideal I (with identity) of N is of the form $l(f)$ for some central idempotent f of N.

Thus $N = l(e) \oplus I$, where the identity of $l(e)$ lies in I and the identity of I lies in $l(e)$, both identities are central idempotents.

Now, N satisfies the acc on right annihilators, by 1.4.4 N satisfies the dcc on left annihilators.

So, we suppose, $l(e_1)$ is a non-zero minimal ideal of N.
Then there exists an ideal I_1(say) such that $N = l(e_1) \oplus I_1$
where e_1 is the identity of I_1 and it is central idempotent of N.

Again, I_1 is a subnear-ring of N and so by 1.4.5, it satisfies acc on right annihilators. Also, I_1 is a d.g.(by Pilz [42], 6.9(e)) and I_1 is weakly regular by 5.2.4.

Thus I_1 is a weakly regular d.g.nr. satisfying the acc on right annihilators in I_1. By the above process,

$I_1 = l_{I_1}(e_2) \oplus I_2$ where $I_2 \subseteq I_1$ and both $l_{I_1}(e_2)$, I_2 are ideals of N by 5.1.7.

Hence $N = l(e_1) \oplus l_{I_1}(e_2) \oplus I_2$, $I_1 \supseteq I_2$.

Continuing the process, we get

$N = l(e_1) \oplus l_{I_1}(e_2) \oplus l_{I_2}(e_3) + \cdots$

and $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$

where each I_i is an ideal of N with identity and is of the form $l(e)$.

But N satisfies dcc on left annihilators and hence the chain $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ must stop after a finite steps (say t).

Therefore, $N = l(e_1) \oplus l_{I_1}(e_2) \oplus \cdots \oplus l_{I_t}(e_{t+1})$.

where each summand has it identity and each of them is distributively generated [by Pilz [42], 6.9(e)] and weakly regular [by 5.2.4].

Since each summand is minimal and hence simple. Thus the result. //

5.4. Strongly prime weakly right Goldie near-rings

In [52], a right Goldie near-ring has been defined as a near-ring with no infinite independent family of right N-subsets of N satisfying the acc on right annihilators.

It is to be noted that some Goldie theorem analogues satisfy this definition. Of course, it is observed that non existence of infinite independent family of right N-subgroups of N is right analogous concept of what has been defined in chapter III as a strictly left Goldie near-ring. In this light, we shall call a near-ring N with acc on right annihilators having no infinite independent family of right N-subgroups of N as a weakly right Goldie near-ring.

In this section, we mainly observe the existence of quasi near-domain structure of a factor near-ring of a left annihilator in a strongly prime weakly right Goldie near-ring.

Throughout the section N will be a strongly prime near-ring (not necessarily with 1) with distributively generated left annihilators.

Let N be with acc on right annihilators and $J = r(T)$ is a maximal right annihilator. Then $M = l(J)$ is distributively generated. Let $M = \langle S \rangle$ where S is a set of distributive elements. So, $S \subseteq M$ and $r(S) \supseteq r(M) = r(l(r(T))) = r(T)$

$\Rightarrow r(S) = r(T) = J$, as J is a maximal right annihilator.

Unless otherwise specified M, J and T will be as above and non-zero throughout the section.

5.4.1. Proposition : The set J is a proper right ideal of N.

Proof: We have already seen that

$J = r(T) = r(M) = r(S)$ where $M = l(J)$ and S is a set of distributive elements with $M = \langle S \rangle$.

Now, $J = r(T) \Rightarrow TJ = (0)$

If $J = N$ then $TN = (0)$

$\Rightarrow T \subseteq l(N) = (0)$ (by 5.1.1.)

$\Rightarrow T = (0)$, a contradiction.

Thus $J \neq N$.

Now let $a, b \in J = r(S)$, then

$s(a-b) = sa - sb$, (s distributive)

$= 0$, $(a, b \in r(S))$

$\Rightarrow S(a-b) = (0)$

$\Rightarrow a-b \in r(S) = J$

$\Rightarrow a-b \in J$.

Next for $j \in J$, $s \in S$, $n \in N$ we have

$s(-n+j+n) = -sn + sj + sn$, (s is distributive)

$= -sn + sn$, $(sj = 0$ for $j \in r(S))$

$= 0$, for all $s \in S$.

$\Rightarrow$ $S(-n + j + n) = (0)$

$\Rightarrow$ $(-n + j + n) \in \mathbf{r}(S) = J$.

Thus J is a normal subgroup of N.

If $n \in N$, $a \in J$ then for $s \in S$,

$s(an) = (sa)n = 0$, for all $s \in S$

$\Rightarrow$ $an \in r(S) = J$.

Thus, J is a right ideal of N.

Therefore, J is a proper right ideal of N. //

5.4.2. Proposition : $M \cap J$ is a proper ideal of M.

Proof : Since each of M and J is a normal subgroup of $(N,+)$, so is also $M \cap J$.

And $J \cap M \subseteq M$ gives that $J \cap M$ is a normal subgroup of M.

Let $x \in J \cap M$, then for $m \in M$

$xm \in J$ (as J is a right ideal of N).

and $xm \in M$ (as M is a left ideal of N).

$\Rightarrow$ $xm \in J \cap M$.

Next, let $x \in J \cap M$; $p, m \in M (= \langle S \rangle)$,

then $s\,(m\,(x+p) - mp)$, for $s \in S$.

$= sm\,(x+p) - smp$, (s is distributive)

$= s(\Sigma \pm s_i)(x+p) - s(\Sigma \pm s_i)\, p, \quad (m \in M,\ s_i \in S)$

$= (\Sigma \pm ss_i)(x + p) - (\Sigma \pm ss_i)p$

$= \pm ss_1(x+p) \pm ss_2(x+p) + \ldots - (\Sigma \pm ss_i)\, p$

$= \pm ss_1p \pm ss_2p \pm \ldots - (\Sigma \pm ss_i)p$, as $ss_ix = 0$ for $x \in r(M)$

$= (\Sigma \pm ss_i)\, p - (\Sigma \pm ss_i)p$

$= 0$, for all $s \in S$.

$\Rightarrow m(x+p) - mp \in r(S) = J$

And clearly, $m(x+p) - mp \in M$

$\Rightarrow m(x+p) - mp \in J \cap M$.

Thus $J \cap M$ is an ideal of M.

If possible, let $J \cap M = M$.

Then $(J \cap M)^2 = (J \cap M)(J \cap M) \subseteq MJ = (0)$

$\Rightarrow M^2 = (0)$

$\Rightarrow M = (0)$, (N-strongly prime), a contradiction.

Hence $J \cap M \subset M$.

Thus $J \cap M$ is a proper ideal of M. //

5.4.3. <u>Definition</u> : A near-ring N is called <u>(left) cancellative</u> if for $x, y, a\ (\neq 0) \in N$, $ax = ay \Rightarrow x = y$.

We have number of examples of near-ring N where $x, y \in N$, $xy = 0,\ x \neq 0 \Rightarrow y = 0$, such as

5.4.4. Example (F(7), Page 340, [42]): N = {0,1,2,3,4} is a near-ring under addition modulo 5 and multiplication defined by the following table.

.	0	1	2	3	4
0	0	0	0	0	0
1	0	1	4	1	4
2	0	2	3	2	3
3	0	3	2	3	2
4	0	4	1	4	1

Table : 5.2.

Here we see that for non-zero $2 \in N$, $2.1 = 2.3 = 2$ but $1 \neq 3$. So, there exists a non-zero $a \in N$ and $x \neq y$ for $x,y \in N$ such that $ax = ay$. In otherwerds, N is not (left) cancellative. //

Now we extend the definition of a near-domain due to Graves [28] to a quasi near-domain.

5.4.5. Definitions : A near-ring is a near-domain [due to Graves] if

(i) for $x,y,a \in N$, $a \neq 0$

$$ax = ay \Rightarrow x = y.$$

(ii) for any $x,y \in N$, $(x \neq 0 \neq y)$,

$$xN \cap yN \neq (0).$$

A near-ring N is called a quasi near-domain if

(i) for $x,y \in N$, $xy = 0$ ($x \neq 0$)

$$\Rightarrow \quad y = 0$$

(ii) for any $x,y \in N$, $(x \neq 0 \neq y)$

$$xN \cap yN \neq (0) .$$

Clearly a near-domain is a quasi near-domain

(for $a \neq 0$, $x \in N$, $ax = 0 \Rightarrow a.x = a.0 \Rightarrow x = 0$)

In what follows, $SM \subseteq S$. In otherwords, S is a right M-subset of M.

5.4.6. Proposition : The near-ring $M/(J \cap M) (= \overline{M})$ is such that for $\overline{m},\overline{n} \in \overline{M}$, $\overline{m}\,\overline{n} = \overline{0}$ ($\overline{m} \neq \overline{0}$) gives $\overline{n} = \overline{0}$.

Proof : Since $\overline{m} \neq \overline{0}$, $m \notin J$ (as $m \in M$, $m \notin J \cap M$)

$$\Rightarrow \quad Sm \neq (0)$$

Now, $(Sm)n = S(mn) = (0)$, since $mn \in J = r(S)$.

$\Rightarrow \quad n \in r(Sm) = r(S) = J$ (as J is maximal and $SM \subseteq S$)

$\Rightarrow \quad n \in J \cap M$

$\Rightarrow \quad \overline{n} = \overline{0}$. //

5.4.7. Lemma : Let for $s,q \in S$, $J \cap (sN + qN) = (0)$ $(s \neq 0 \neq q)$ such that $sN \cap qN = (0)$.

Then there exists $s_1 \in S$ ($\subseteq M$) such that $s_1qN \cap s_1sN = (0)$ and s_1sN, $s_1qN \subseteq sN$.

Proof : First we prove that $sNM \neq (0)$.

For, $sNM = (0)$

$\Rightarrow sN \subseteq l(M) = (0)$, (by 5.1.2)

$\Rightarrow s \in l(N) = (0)$, (by 5.1.1)

$\Rightarrow s = 0$, not true

$\Rightarrow sNM \neq (0)$.

Let snm ($\neq 0$) $\in sNM$ where $n \in N$, $m \in M$ then

$sx \in sNM$ where $x = nm \in M$.

$\Rightarrow sx \in SM \subseteq S$

Let $s_1 = sx$ ($\in S$)

Here we note that $s_1sN \neq (0)$.

For, $s_1sN = (0)$

$\Rightarrow sN \subseteq r(s_1) = J$ ($= r(S)$)

$\Rightarrow sN \subseteq (sN + qN) \cap J = (0)$

$\Rightarrow s \in l(N) = (0)$

$\Rightarrow s = 0$, not true.

Thus, $s_1sN \neq (0)$.

Moreover, $s_1 (= sx) \in sN$

$\Rightarrow s_1 sN = (sx)sN = s(xsN) \subseteq sN.$

and $s_1 qN = sxqN \subseteq sN$

Therefore, $s_1 qN, s_1 sN \subseteq sN$

Next we prove that $s_1 qN \cap s_1 sN = (0)$

Suppose, $s_1 qx = s_1 sy \in s_1 qN \cap s_1 sN, (x, y \in N)$

$\Rightarrow s_1(sy - qx) = 0, \quad s_1 \in S$

$\Rightarrow sy - qx \in r(s_1) = J$

Now, $sy - qx = sy + q(-x) \in sN + qN, \ (q \in S)$

$\Rightarrow sy - qx \in J \cap (sN + qN) = (0)$

$\Rightarrow sy = qx \in sN \cap qN = (0)$

$\Rightarrow s_1 qx = s_1 sy = 0$

$\Rightarrow s_1 qN \cap s_1 sN = (0)$. //

5.4.8. Lemma : Let $A, B \subseteq sN$, $s \in S$ such that $sN \cap qN = (0)$ for $q \in S$.

Then $(qN + A) \cap B \subseteq A \cap B$.

Proof : Let $\alpha = b = qx + a \in (qN + A) \cap B$ where $a \in A$, $b \in B$, $x \in N$.

Then $a = sy$, $b = sz$, for $y, z \in N$.

$\Rightarrow\ sz = qx + sy$

$\Rightarrow\ qx = s(z-y) \in qN \cap sN = (0)$

$\Rightarrow\ b = a \in A \cap B$.

Thus, $(qN + A) \cap B \subseteq A \cap B$. //

5.4.9. Lemma : Let $J \cap (sN + qN) = (0)$, $s, q \in S$ such that $sN \cap qN = (0)$ (for $s \neq 0 \neq q$).

Then there exists $s_1 \in S\ (\subseteq M)$ such that $\{qN, s_1qN, s_1sN\}$ is an independent family of right N-subgroups of N with s_1qN, $s_1sN \subseteq sN$.

Proof : By 5.4.7, there exists $s_1 \in S\ (\subseteq M)$ such that $s_1qN \cap s_1sN = (0)$ and $s_1sN, s_1qN \subseteq sN$.

So, $(s_1qN + s_1sN) \cap qN \subseteq (sN + sN) \cap qN$

$\subseteq sN \cap qN = (0)$

$\Rightarrow\ (s_1qN + s_1sN) \cap qN = (0)$.

Next, $(qN + s_1sN) \cap s_1qN \subseteq s_1sN \cap s_1qN$ (by 5.4.8.)

$\Rightarrow\ (qN + s_1sN) \cap s_1qN = (0)$

Also, $(qN + s_1qN) \cap s_1sN \subseteq s_1qN \cap s_1sN$ (as above)

$\Rightarrow\ (qN + s_1qN) \cap s_1sN = (0)$.

Thus $\{qN, s_1qN, s_1sN\}$ is an independent family of right N-subgroups of N. //

5.4.10. Theorem : Let N be a strongly prime weakly right Goldie near-ring such that $sN \cap qN = (0)$ for non-zero $s,q \in S$ ($\subseteq M$).

Then $J \cap (qN + sN) \neq (0)$.

Proof : Suppose, $J \cap (qN + sN) = (0)$.

Then by 5.4.9 there exists $s_1 \in S$ such that $\{qN, s_1qN, s_1sN\}$ is an independent family with $s_1qN, s_1sN \subseteq sN$.

Since $J \cap (s_1qN + s_1sN) \subseteq J \cap (sN + sN)$

$$\subseteq J \cap sN \subseteq J \cap (sN + qN) = (0)$$

$\Rightarrow \quad J \cap (s_1qN + s_1sN) = (0)$

Following the same argument with $s_1q, s_1s \in S$ there exists $s_2 \in S \subseteq M$ such that $\{s_1qN, s_2s_1qN, s_2s_1sN\}$ is an independent family with $s_2s_1qN, s_2s_1sN \subseteq s_1sN \subseteq sN$.

We claim that $\{qN, s_1qN, s_2s_1qN\}$ is an independent family.

Here $s_1qN, s_2s_1sN \subseteq sN$, $qN \cap s_1qN = (0)$.

$(qN + s_1qN) \cap s_2s_1qN \subseteq s_1qN \cap s_2s_1qN$ (by 5.4.8.)

$\Rightarrow \quad (qN + s_1qN) \cap s_2s_1qN = (0)$

$\Rightarrow \quad (qN + s_1qN)$

And $(qN + s_2s_1qN) \cap s_1qN \subseteq s_2s_1qN \cap s_1qN$ (by 5.4.8)

$= (0)$

Also, $(s_1qN + s_2s_1qN) \cap qN \subseteq (sN + sN) \cap qN$

$\subseteq sN \cap qN = (0)$

Thus $\{qN, s_1qN, s_2s_1qN\}$ is an independent family.

In like manner, we get an independent family of right N-subgroups of N, viz. $\{q,N, s_1qN, s_2s_1qN, s_3s_2s_1q\ N, \ldots\}$ which is an infinite family. And this contradicts the weakly right Goldie character of N.

Therefore, $J \cap (qN + sN) \neq (0)$. //

5.4.11. Theorem : Let N be a strongly prime weakly right Goldie near-ring.

Then $M/J\cap M$ is a quasi near-domain.

Proof : By 5.4.6, for $\bar{m},\bar{n} \in \bar{M}\ (= M/(J\cap M))$, $\bar{m} \neq \bar{0}$, $\bar{m}\bar{n} = \bar{0} \Rightarrow \bar{n} = \bar{0}$.

Next, let $\bar{s},\bar{q} \in \bar{M}$, $s,q \in S$ ($\bar{s} \neq \bar{0} \neq \bar{q}$), we show that $\bar{s}\,\bar{M} \cap \bar{q}\,\bar{M} \neq (\bar{0})$.

(I) Let $sN \cap qN \neq (0)$, then there exists

$c = sx = qy\ (\neq 0)$, for some $x,y \in N$.

If $cM = (0)$ then $c \in l(M) = (0)$, by 5.1.2, a contradiction. So $cM \neq (0)$ which gives an element $p \in M$ such that $cp \neq 0$

Now, $cp = sxp = qyp$

$$\Rightarrow \bar{s}\,\overline{xp} = \bar{q}\,\overline{yp} = \overline{cp}$$

And, $\overline{cp} = \bar{0} \Rightarrow \bar{s}\,\overline{xp} = \bar{q}\,\overline{yp} = \bar{0}$

$\Rightarrow \overline{xp} = \bar{0}$, (for $\bar{s} \neq \bar{0}$) (by 5.4.6.)

$\Rightarrow xp \in J \cap M$

$\Rightarrow xp \in J = r(S)$.

$\Rightarrow Sxp = (0)$

$\Rightarrow sxp = 0$, for all $s \in S$

$\Rightarrow cp = 0$, not true.

Therefore $\overline{cp} \neq \bar{0}$. In otherwords,

$$\bar{s}\,(\overline{xp}) = \bar{q}(\overline{yp}) \neq \bar{0}$$

Thus, $\bar{s}\,\bar{M} \cap \bar{q}\,\bar{M} \neq (\bar{0})$ as $\overline{xp}, \overline{yp} \in \bar{M}$.

(II) Suppose $sN \cap qN = (0)$ for $s, q \in S$ ($\subseteq M$).

Then by 5.4.10, $J \cap (qN + sN) \neq (0)$.

Hence there exists $c = qx + sy$ ($c \neq 0$) for some $x, y \in N$, $c \in J$.

Now, $cM \neq (0)$ as before.

So there exists a $w \varepsilon M$ such that $cw \neq 0$.

Then, $cw = (qx+sy)w = qxw + syw$

Since $c \varepsilon J$, $cw \varepsilon J$ (as J is a right ideal by 5.4.1.) and $cw \varepsilon M$ (as M is a left ideal).

$\Rightarrow \quad cw \varepsilon J \cap M$

$\Rightarrow \quad \bar{c}\,\bar{w} = \bar{0}$

$\Rightarrow \quad (\overline{qx + sy})\,\bar{w} = \bar{0}$

$\Rightarrow \quad \bar{q}\,\bar{x}\,\bar{w} + \bar{s}\,\bar{y}\,\bar{w} = \bar{0}$

$\Rightarrow \quad \bar{q}\,\overline{xw} = \bar{s}\,(-\overline{yw})$, (as $\bar{s}$ distributive)

And, $\bar{q}\,\overline{xw} = \bar{0} \Rightarrow \overline{xw} = \bar{0}$ (for $\bar{q} \neq \bar{0}$)

$\Rightarrow \overline{yw} = \bar{0}$ (for $\bar{s} \neq \bar{0}$)

These give us $xw, yw \varepsilon J$.

$\Rightarrow \quad xw, yw \varepsilon r(S)$.

$\Rightarrow \quad qxw, syw = 0$ for $q, s \varepsilon S(\subseteq M)$.

$\Rightarrow \quad cw = qxw + syw = 0$, contradiction.

Thus $\bar{q}\,\overline{xw} = \bar{s}(-\overline{yw}) \neq \bar{0}$

So, $\bar{q}\,\bar{M} \cap \bar{s}\,\bar{M} \neq (\bar{0})$

Finally, let any $\bar{m}, \bar{m}_1 \varepsilon \bar{M}$. Then

$\overline{m} = \Sigma \pm \overline{s}_i \quad (s_i \in S)$

and $\overline{m}_1 = \Sigma \pm \overline{q}_i \quad (q_i \in S)$

Therefore, $\overline{m}\,\overline{M} = (\Sigma \pm \overline{s}_i)\,\overline{M} \supseteq \overline{s}_i\,\overline{M}$

and $\overline{m}_1\overline{M} = (\Sigma \pm \overline{q}_i)\,\overline{M} \supseteq \overline{q}_i\overline{M}$, for $s_i, q_i \in S$.

And by what we have proved above, we get

$$\overline{s}_i\overline{M} \cap \overline{q}_i\overline{M} \neq (\overline{0})$$

So, $\overline{m}\,\overline{M} \cap \overline{m}_1\,\overline{M} \supseteq \overline{s}_i\,\overline{M} \cap \overline{q}_i\,\overline{M} \neq (\overline{0})$ which gives,

$\overline{m}\,\overline{M} \cap \overline{m}_1\,\overline{M} \neq (\overline{0})$.

Hence $M/_{(J \cap M)}$ is a quasi near-domain. //

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