2016

MATHEMATICS

(Major)

Paper : 5.1

(Real and Complex Analysis)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : 1×7=7

(a) State a sufficient condition for the continuity of a real-valued function of two variables.

(b) Give an example of a real-valued function which is bounded but not Riemann integrable.

(c) A real-valued function f is defined on $[a, b]$ having a singular point in its domain. State whether f is Riemann integrable or not.

(Turn Over)

(d) A function $f(z) = u(x, y) + iv(x, y)$ is defined such that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

State whether f is analytic or not.

(e) Evaluate

$$\frac{1}{2\pi i} \oint_C \frac{\cos \pi z}{z^2 - 9} dz$$

where C is a closed rectangle with vertices at $z = 2 \pm i, -2 \pm i$.

(f) State Cauchy's integral formula.

(g) Define conformal mapping.

2. Answer the following questions : 2×4=8

(a) Discuss the continuity of the following function at (0, 0) :

$$f(x, y) = \frac{xy^3}{x^2 + y^6}, \quad (x, y) \neq (0, 0)$$
$$= 0, \quad (x, y) = (0, 0)$$

(b) Show that the integral

$$\int_0^1 x^{m-1} e^{-x} dx$$

is convergent for $m > 0$.

(Continued)

(c) Prove that if $w = f(z) = u + iv$ is analytic in a region R, then

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i \frac{\partial w}{\partial y}$$

where u, v are functions of two variables x, y.

(d) Find the fixed points of the transformation $w = \dfrac{2z-5}{z+4}$.

3. Answer any *three* parts : 5×3=15

(a) If

$$u = \cos x, \; v = \sin x \cos y$$
$$w = \sin x \sin y \cos z$$

then show that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-i)^3 \sin^3 x \sin^2 y \sin z$$

The symbols have their usual meanings.

(b) Prove that if f is a bounded function on $[a, b]$, then to every $\varepsilon > 0$, there corresponds $\delta > 0$ such that

$$U(p, f) < \int_a^b f dx + \varepsilon$$

The symbols have their usual meanings.

(Turn Over)

(c) Show that the integral

$$\int_0^{\pi/2} \log \sin x \, dx$$

is convergent. Hence evaluate it.

(d) Prove that if $f(z)$ and $g(z)$ are analytic at z_0 and $f(z_0) = g(z_0) = 0$ but $g'(z_0) \neq 0$, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

(e) Prove that if $f(z)$ is integrable along a curve C having finite length L and if there exists a positive number M such that $|f(z)| \leq M$ on C, then

$$\left| \int_C f(z) \, dz \right| \leq ML$$

4. Answer *either (a) or (b)* :

(a) (i) If v is a function of two variables x and y, and $x = r\cos\theta$, $y = r\sin\theta$, then prove that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r}\frac{\partial v}{\partial r}$$

(Continued)

(ii) Find the shortest distance from the origin to the hyperbola

$$x^2 + 8xy + 7y^2 = 225, \; z = 0$$ 5

(b) (i) Verify the convergence of the integral

$$\int_0^\infty \frac{x \tan^{-1} x}{(1 + x^4)^{1/3}} dx$$ 5

(ii) Find the value of p such that

$$\int_1^\infty \frac{\sin x}{x^p} dx$$

converges absolutely. 5

5. Answer *either (a) or (b)* :

(a) (i) Prove that if a function f is Riemann integrable on $[a, b]$, then f^2 is also Riemann integrable on $[a, b]$. 5

(ii) A function f is defined on $[-1, 1]$ as follows :

$$f(x) = 1, \; x \neq 0$$
$$= 0, \; x = 0$$

Show that f is integrable on $[-1, 1]$ and calculate its value. 5

(Turn Over)

(b) (i) Show that the function f defined as follows

$$f(x) = \frac{1}{2^n}, \text{ when } \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n} \; (n = 0, 1, 2, \cdots)$$
$$= 0 \;, \; x = 0$$

is integrable on [0, 1]. Also evaluate

$$\int_0^1 f\,dx$$ 5

(ii) If f and g are both differentiable on $[a, b]$ and if f', g' are both integrable on $[a, b]$, then show that

$$\int_a^b f(x)\,g'(x)\,dx = [f(x)\,g(x)]_a^b - \int_a^b g(x) f'(x)\,dx$$ 5

6. Answer *either* (a) *or* (b) :

(a) (i) If

$$u_1(x, y) = \frac{\partial u}{\partial x}$$

and $u_2(x, y) = \dfrac{\partial u}{\partial y}$

then prove that

$$f'(z) = u_1(z, 0) - iu_2(z, 0)$$ 5

(ii) Prove that $\dfrac{d}{dz}(z^2\,\bar{z})$ does not exist anywhere. 5

(Continued)

(b) (i) Evaluate

$$\oint_C \bar{z}^2\,dz$$

around the circle $|z-1| = 1$. 5

(ii) Find a bilinear transformation which maps $z = 0, -i, -1$ into $w = i, 1, 0$ respectively. 5

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2016

MATHEMATICS

(Major)

Paper : 5.3

(Spherical Trigonometry and Astronomy)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer all the questions : 1×7=7

   (a) How many great circles can be drawn through two given points?

   (b) Define spherical triangle.

   (c) Explain what is meant by 'parallel of latitudes'.

   (d) At the position of transit of a star, what is the property of altitude and its zenith?

   (e) What is the relation between the linear velocity and perpendicular distance from the centre upon the tangent to the path of a central orbit?

(f) Stating the physical situation, define annular solar eclipse.

(g) State the position of the sun which is known as summer solstice.

2. Answer *all* the following questions : 2×4=8

(a) What are meant by ecliptic limits? Explain.

(b) Show that the sum of the three sides of a spherical triangle is less than the circumference of a great circle.

(c) Prove that for a right spherical triangle where $C = \pi/2$, $\cos A = \tan b \cot c$.

(d) If T is the orbital period of a planet, show that a small increment Δa in the semi-axis a will produce an increase $\frac{3T\Delta a}{2a}$ in the period.

3. Answer any *three* parts of the following : 5×3=15

(a) Explain about the dynamical significance of the Kepler's laws.

(b) Show that the velocity of a planet in its orbit has got two constant components, one perpendicular to the radius vector and the other perpendicular to the major axis.

(c) Distinguish between geocentric parallax and annual parallax of a star. Determine the effects of annual or stellar parallax on right ascension and declination.

(d) Explain with the help of neat diagrams the coordinate systems of celestial sphere.

(e) If a is the sun's altitude in the prime vertical at a place of latitude ϕ and L is its longitude, prove that

$$\phi = \sin^{-1}(\sin L \sin \varepsilon \operatorname{cosec} a)$$

4. In a spherical triangle, prove that

$$\cos a \cos C = \sin a \cot b - \sin C \cot B$$

Also prove that, if a be the side of an equilateral spherical triangle and a' that of its polar triangle, then

$$2\cos\frac{a}{2}\cos\frac{a'}{2} = 1$$

6+4=10

5. Show that the mathematical condition for lunar eclipse to be possible of some kind is

$$\xi < D(1 - 2q\cos i + q^2)^{1/2} \operatorname{cosec} i$$

where $D = \alpha \pm \gamma_{☾}$ for partial and total eclipse respectively,

$$q = \frac{\theta}{\phi} = \frac{\text{rate of increase of sun's longitude}}{\text{moon's angular velocity in its orbit}}$$

the other symbols carry their usual meanings. 10

6. Discuss the effects of refraction on sunrise and sunset. 10

Or

(a) Show that the retardation due to parallax in the time of rising of an object of geocentric parallax p seconds of arc and of declination δ is

$$\frac{1}{15}\frac{p}{\sqrt{(\cos^2\phi - \sin^2\delta)}} \text{ seconds,}$$

ϕ being the latitude of the place.

(b) If S is the semi-vertical angle of the tangent cone to the moon from the earth's centre when the moon's horizontal parallax is E and if S', P' be another similar parts, prove that the earth being supposed spherical

$$\frac{\sin S}{\sin S'} = \frac{\sin P}{\sin P'}$$

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2016

MATHEMATICS

(Major)

Paper : 5.4

(Rigid Dynamics)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : 1×7=7

   (a) Write down the moment of inertia of a circular ring of mass M and radius a about an axis through its centre and perpendicular to its plane.

(b) Write down the radius of gyration of a spherical shell of mass M and radius a about a diameter.

(c) A particle of mass 4 units is placed at the point $(-1, -1, 1)$. What is the product of inertia of the particle about OX-OY?

(d) State the perpendicular axes theorem on moments of inertia.

(e) A rigid body moves freely in space. What is the degree of freedom of the body?

(f) Define the centre of suspension of a compound pendulum.

(g) Define conservative system.

(Continued

2\. Answer the following questions : $2 \times 4 = 8$

(a) A rigid body consists of 3 particles of masses 2, 1, 4 located at $(1, -1, 1)$, $(2, 0, 2)$ and $(-1, 1, 0)$ respectively. Find the moments of inertia about the x, y and z axes.

(b) A rigid body of mass 2 units rotates with angular velocity $\vec{\omega} = (1, 1, -1)$ and has the angular momentum $\vec{\Omega} = (2, 3, -1)$. Find the kinetic energy of the body.

(c) A body with one point fixed rotates with angular velocity $(0, 0, 2)$. Find the magnitude of the velocity of a particle of mass m of the body located at the point $(3, -4, 1)$.

(d) A particle moves under the influence of central force field $f(r)\vec{r}$, where $r = |\vec{r}|$, $\vec{r}$ being the position vector of the particle relative to the centre of force O. Show that the angular momentum of the particle about O is constant.

(Turn Over)

3. Answer the following questions : 5×3=15

(a) Find the moment of inertia of a uniform triangular lamina about one side.

Or

Find the moment of inertia of a solid sphere of radius a and mass M about a diameter.

(b) State and prove d'Alembert's principle.

Or

Show that the motion of a body about its centre of inertia is the same as it would be if the centre of inertia were fixed and the same forces acted on the body.

(c) Show that a uniform rod of mass m is kinetically equivalent to three particles rigidly connected and situated one at each end of the rod and its middle point, the masses of the particles being $\frac{1}{6}m$, $\frac{1}{6}m$ and $\frac{2}{3}m$.

(Continued

4. A rod of length $2a$ revolves with uniform angular velocity ω about a vertical axis through a smooth joint at one extremity of the rod so that it describes a cone of semi-vertical angle α, show that

$$\omega^2 = \frac{3g}{4a\cos\alpha}$$

Prove also that the direction of reaction at the hinge makes with the vertical an angle

$$\tan^{-1}\left(\frac{3}{4}\tan\alpha\right)$$ 10

Or

(a) A plank of mass M is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon and a man of mass M', starting from the upper end walks down the plank so that it does not move. Show that he gets to the other end in time

$$\sqrt{\frac{2M'a}{(M+M')g\sin\alpha}}$$

where a is the length of the plank. 6

(Turn Over)

(b) Using d'Alembert's principle, derive the general equations of motion of a rigid body. 4

5. (a) An elliptical lamina is such that when it swings about one latus rectum as a horizontal axis, the other latus rectum passes through the centre of oscillation. Prove that the eccentricity is $\frac{1}{2}$. 5

(b) Set up the Lagrangian for a simple pendulum and obtain an equation describing its motion. 5

6. Write down the equations of motion of a rigid body in two dimensions under impulsive forces. Two equal uniform rods, AB and AC, are freely jointed at A, and are placed on a smooth table so as to be at right angles. The rod AC is struck by a blow at C in a direction perpendicular to itself. Show that the resulting velocities of the middle points of AB and AC are in the ratio 2 : 7. 3+7=10

(Continued)

Or

AB and CD are two equal similar rods connected by a string BC; AB, BC and CD form three sides of the square. The point A of the rod AB is struck a blow in a direction perpendicular to the rod. Show that the initial velocity of A is seven times that of D. 10

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2016

MATHEMATICS

(Major)

Paper : 5.5

(Probability)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions as directed : 1×8=8

(*a*) If A and B are mutually exclusive, what will be the modified statement of $P(A \cup B) = P(A) + P(B) - P(A \cap B)$?

(b) If a non-negative real valued function f is the probability density function of some continuous random variable, then what is the value of $\int_{-\infty}^{\infty} f(x)\,dx$?

(c) What is meant by mathematical expectation of a random variable?

(d) For a Bernoulli random variable X with $P(X = 0) = 1 - p$ and $P(X = 1) = p$, write $E(X)$ and $V(X)$ in terms of p.

(e) Can the probabilities of three mutually exclusive events A, B, C as given by $P(A) = \frac{2}{3}$, $P(B) = \frac{1}{4}$ and $P(C) = \frac{1}{6}$ be correct? If not, give reasons.

(f) Test the validity of the following probability distribution :

x	−1	0	1
$p(x)$	0·4	0·4	0·3

(g) Under what condition $\text{cov}(X, Y) = 0$?

(h) Choose the correct option for binomial distribution.

(i) Variance = Mean

(ii) Variance > Mean

(iii) Variance < Mean

2. Answer the following questions : 3×4=12

(a) A speaks the truth in 75% cases and B in 80%. In what percentages of cases are they likely to contradict each other while narrating the same incident?

(b) Define probability mass function and probability density function for a random variable X.

(c) The second moment about any point (a) is minimum when taken about the mean (μ), i.e.,

$$E(X-a)^2 \geq E(X-\mu)^2$$

Prove or disprove the above statement.

(d) State the three Poisson's postulates.

3. Answer any *two* parts from the following questions : 5×2=10

(a) What is meant by partition of a sample space S? If $H_i = (i = 1, 2, \cdots, n)$ is a partition of the sample space S, then for any event A, prove that

$$P(H_i / A) = \frac{P(H_i) P(A/H_i)}{\sum_{i=1}^{n} P(H_i) P(A/H_i)}$$

(b) Three machines A, B and C produce respectively 60%, 30% and 10% of the total number of items of a factory. The percentage of defective output of these machines are respectively 2%, 3% and 4%. An item is selected at random and is found to be defective. Find the probability that the item was produced by machine C.

(c) A bag contains 5 balls and it is known how many of these are white. Two balls are drawn and are found to be white. What is the probability that all are white?

(Continued)

4. Answer any *two* parts from the following questions : 5×2=10

(a) Let F be the distribution function of a two-dimensional random variable (X, Y). If $a < b$ and $c < d$, then show that

$$P(a < X \le b, c < Y \le d) = F(b, d) + F(a, c) - F(a, d) - F(b, c)$$

(b) If X is a discrete random variable having probability mass function

Mass point	0	1	2	3	4	5	6	7
$P(X = x)$	0	k	$2k$	$3k$	$4k$	k^2	$2k^2$	$7k^2 + k$

Determine (i) k, (ii) $P(X < 6)$ and (iii) $P(X \ge 6)$. 2+2+1=5

(c) The probability density function of a two-dimensional random variable (X, Y) is given by

$$f(x, y) = x + y, \quad 0 < x + y < 1$$
$$= 0, \quad \text{elsewhere}$$

Evaluate $P(X < \frac{1}{2}, Y > \frac{1}{4})$.

(Turn Over)

5. Answer any *two* parts from the following questions : 5×2=10

(a) From a bag containing 4 white and 6 red balls, three balls are drawn at random. Find the expected number of white balls drawn.

(b) For any two independent random variables X and Y, for which $E(X)$ and $E(Y)$ exist, show that

$$E(XY) = E(X)E(Y)$$

(c) The probability density function of a continuous bivariate distribution is given by the joint density function

$$f(x, y) = x + y, \quad 0 < x < 1, \; 0 < y < 1$$
$$= 0 \quad , \text{ elsewhere}$$

Find $E(X)$, $E(Y)$, $\text{var}(X)$, $\text{var}(Y)$ and $E(XY)$.

6. Answer any *two* parts from the following questions : 5×2=10

(a) If X is a Poisson distributed random variable with parameter μ, then show that $E(X) = \mu$ and $\text{var}(X) = \mu$.

(Continued

(b) Show that normal distribution may be regarded as a limiting case of Poisson's distribution as the parameter $m \to \infty$.

(c) Define binomial distribution. What is the probability of guessing correctly at least six of the ten answers in a True-False objective test?

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2016

MATHEMATICS

(Major)

Paper : 5.6

(Optimization Theory)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Fill in the blanks : 1×7=7

(a) The set of all feasible solutions of an LPP is a _____ set.

(b) A system of m linear equations in n unknowns has at most _____ basic solutions.

(c) A BFS is a basic solution whose variables are _____.

(d) A BFS having more than $(n-m)$ variables 0, is named as _____, where the system consists of m linear equations in n unknowns.

(e) The objective function of an LPP is optimum at a _____ solution.

(f) If the jth primal variable x_j is unrestricted in sign, then in the dual problem, the jth constraint is an _____.

(g) In a balanced transportation problem with m sources and n destinations, the number of constraint equations is _____.

2. Answer the following questions : 2×4=8

(a) What do you mean by an LPP?

(b) The f.s. $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $z = 6$ to the system

$$x_1 + x_2 + x_3 = 2,$$
$$x_1 - x_2 + x_3 = 2,$$
$$\text{Min } Z = 2x_1 + 3x_2 + 4x_3,$$
$$x_i \geq 0 \quad i = 1, 2, 3$$

is not basic. Justify.

(Continued)

(c) Define convex set.

(d) Which of the points (0, 0), (0, 1), (1, 2), (1, 1), (4, 0) is an interior point of the convex hull of the above points?

3. Answer any *three* parts of the following : 5×3=15

(a) Upon completing the construction of his house, Mr. Somani discovered that 100 sq. ft. of plywood scrap and 80 sq. ft. of white pine scrap are in usable form for construction of tables and bookcases. It takes 16 sq. ft. of plywood and 2 sq. ft. of white pine to make a table and 12 sq. ft. of plywood and 16 sq. ft. of white pine for a bookcase. By selling the finished products to a local furniture store, Mr. Somani can realize a profit of ₹ 25 on each table and ₹ 20 on each bookcase. How may he most profitably use the left-over wood? Use graphical method to solve the problem.

(b) If S and T are any two convex sets in R^n, then for all scalars α, β, prove that the set $\alpha S + \beta T$ is also a convex set.

(Turn Over)

(c) If x is any f.s. to the primal

$$\text{Min } Z_p = \overline{C}\overline{x}$$

subject to $A\overline{x} \geq \overline{b}$, $\overline{x} \geq 0$

and w is any f.s. to the dual

$$\text{Max } Z_D = \overline{b}' w$$

subject to $A'\overline{w} \leq \overline{C}'$, $\overline{w} \geq 0$

then show that $Z_p \geq Z_D$.

(d) Give the dual of the following and solve :

$$\text{Min } Z = x_1 + 2x_2$$

subject to

$$2x_1 + x_2 \geq 4$$
$$x_1 + 7x_2 \geq 7$$
$$x_1, x_2 \geq 0$$

4. Solve the following LPP by simplex method : 10

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3$$

subject to

$$2x_1 + 3x_2 \leq 8$$
$$3x_1 + 2x_2 + 4x_3 \leq 15$$
$$2x_2 + 5x_3 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$

(Continued)

Or

Find all BFS for the equations 10

$$2x_1 + 6x_2 + 2x_3 + x_4 = 3$$
$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$$

5. Solve the following LPP by using the big M-method : 10

$$\text{Max } Z = x_1 + 2x_2 + 3x_3 - x_4$$

subject to

$$x_1 + x_2 + 3x_3 = 15$$
$$2x_1 + x_2 + 5x_3 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Or

A diet-conscious housewife wishes to ensure certain minimum intake of vitamins A, B and C for the family. The minimum daily (quantity) needs of vitamins A, B and C for the family are respectively 30, 20 and 16 units. For the supply of these minimum vitamin requirements, the housewife relies

(Turn Over

on two fresh foods. The first one provides 7, 5, 2 units of the three vitamins per gram respectively and the second one provides 2, 4, 8 units of the same three vitamins per gram of the foodstuff respectively. The first foodstuff costs ₹3 per gram and the second foodstuff costs ₹2 per gram.

(a) Formulate the underline LP problem.

(b) Write the dual problem.

(c) Solve the dual problem by using simplex method and also write the solution of the primal from the final simplex table of the dual. 4+2+4=10

6. Find a solution of the following transportation problem which will minimize the total cost : 10

$D \rightarrow$ $O \downarrow$	D_1	D_2	D_3	D_4	Available a_i
	1	2	1	4	30
	3	3	2	1	50
	4	2	5	9	20
Requirement b_j	20	40	30	10	

Or

A company is faced with the problem of assigning six different machines to five different jobs. The costs are estimated as follows (hundreds of rupees) :

Jobs → Machines ↓	J_1	J_2	J_3	J_4	J_5
M_1	2·5	5	1	6	1
M_2	2	5	1·5	7	3
M_3	3	6·5	2	8	3
M_4	3·5	7	2	9	4·5
M_5	4	7	3	9	6
M_6	6	9	5	10	6

Solve the problem assuming that the objective is to minimize the total cost. 10

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