2 0 1 5

MATHEMATICS

(Major)

Paper : 5.2

(Topology)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : 1×7=7

(a) Describe the open sphere of unit radius about (0, 0) for the following metric on $\mathbb{R}^2$:

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2|,$$
$$x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2$$

(b) Consider $\mathbb{R}$ with the usual metric. Find the derived set of each of the following subsets of $\mathbb{R}$:

$$A =]0, 1[$$
$$B = \left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}$$
$$C = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$$

(Turn Over)

(c) A finite set in any metric space has no limit point.
Justify whether it is true or false.

(d) Give an example to show that the union of two topologies on a set may not be again a topology.

(e) Let $\mathscr{T}$ be the topology on $\mathbb{N}$ which consists of ϕ and all subsets of the form $G_m = \{m, m+1, m+2, ...\}$, $m \in \mathbb{N}$. What are the open sets containing 5?

(f) Consider the topology $\mathscr{T} = \{\phi, \{a\}, \{b, c\}, X\}$ on $X = \{a, b, c\}$ and the topology $\mathscr{U} = \{\phi, Y, \{r\}, \{p, q\}\}$ on $Y = \{p, q, r\}$. Let $f : (X, \mathscr{T}) \to (Y, \mathscr{U})$ be a mapping defined by $f(a) = p$, $f(b) = q$ and $f(c) = r$. State whether f is a homeomorphism or not.

(g) Define an inner product space and give an example.

2. Answer the following questions : $2\times4=8$

(a) Give an example to show that in a metric space union of an infinite number of closed sets may not be a closed set.

(Continued)

(b) Let $X = \{1, 2, 3, 4\}$ and $A = \{\{1, 2\}, \{2, 4\}, \{3\}\}$. Determine the topology on X generated by A as a subbase and hence determine the base for this topology.

(c) Every inner product space is a normed linear space.
Justify whether it is true or false.

(d) Let $(X, \|\cdot\|)$ be a normed linear space and $x_n \to x$ and $y_n \to y$ in X. Show that

$$x_n + y_n \to x + y$$

3. Answer the following questions : $5\times3=15$

(a) Let (X, d) be a metric space and A be a subset of X. If x is a limit point of A, prove that there exists a sequence $\langle a_n \rangle$ of points of A, all distinct from x, which converges to x.

(b) Let $(X, \mathscr{T})$ be a topological space and $A \subset X$. Prove that the union of A and the set of its accumulation points is closed.

Or

Let $(X, \mathscr{T})$ be a topological space and $\langle f_n \rangle$ be a sequence of complex valued functions defined on X which converges uniformly to a function f defined on X. Prove that if all f_n's are continuous, then f is also continuous.

(Turn Over)

(c) Show that $\mathbb{R}^n$ is a normed linear space with some suitable norm.

Or

If x and y are any two vectors in an inner product space $(X, \langle \cdot, \cdot \rangle)$, then prove that $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$.

4. Answer the following questions : 10×3=30

(a) Let $C[a, b]$ be the set of all real valued continuous functions defined on $[a, b]$. For $f, g \in C[a, b]$, define

$$d(f, g) = \sup_{x \in [a, b]} |f(x) - g(x)|$$

Show that with respect to this metric, $C[a, b]$ is a complete metric space. 10

Or

Let (X, d) be a metric space. Show that the mapping $d_1 : X \times X \to \mathbb{R}$ defined by

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric on X and that d and d_1 are equivalent. 10

(b) State and prove Baire's Category theorem for metric spaces. 10

Or

Let (X, d) be a metric space and $x_0 \in X$ be fixed. Show that the real valued function $f_{x_0}(x) = d(x, x_0)$, $x \in X$ is continuous. Is it uniformly continuous? Let (Y, ρ) be another metric space and $f : X \to Y$ be a mapping. Prove that f is continuous if and only if the inverse image of every open set in Y is an open set in X. 2+1+7=10

(c) Prove that a metric space is sequentially compact if and only if it has the Bolzano-Weierstrass property. 10

Or

Define disconnected metric space and give an example. Let (X, d) be a metric space and A be a connected subset of X such that $A \subseteq B \subseteq \bar{A}$. Prove that B is connected and hence deduce that $\bar{A}$ is connected. 2+8=10

★★★

2015

MATHEMATICS

(Major)

Paper : 5.3

(Spherical Trigonometry and Astronomy)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer *all* the questions : 1×7=7

   (a) Define celestial horizon of celestial sphere and name the poles of it.

   (b) What do you mean by great circle and what is its connection in determining celestial coordinates?

   (c) Before postulating the Kepler's planetary laws, who was the immediate predecessor of Kepler?

   (d) Mention a similar property of a small circle and a great circle for a sphere.

(Turn Over)

(e) Define the polar triangle of a spherical triangle.

(f) What is the difference in totality of lunar and solar eclipses?

(g) State the Cassini's hypothesis of refraction.

2. Answer the following questions : 2×4=

(a) Mention the physical situations when a lunar eclipse can occur.

(b) Show that in a spherical triangle

$$\pi < A + B + C < 3\pi$$

(c) Show that the zenith distance of a star is the compliment of the altitude.

(d) Defining sidereal time and solar time, distinguish them clearly.

3. Answer any *three* of the following : 5×3=

(a) If V_1 and V_2 are the linear velocities of a planet at perihelion and aphelion respectively, prove that

$$V_1 : V_2 = (1+e) : (1-e)$$

(b) If in a spherical triangle ABC, $A = a$, show that B and b are equal or supplemental.

(c) Show that the right ascension (RA) α and declination δ of the sun is connected by the relation

$$\tan\delta = \tan\varepsilon \sin\alpha$$

(d) Prove that if at a certain instant the declination of a star is unaffected by refraction, the azimuth of the star is then maximum and the star culminates between the pole and the zenith.

4. In any spherical triangle, prove that

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

Hence prove that

$$\frac{\sin(A+B)}{\sin C} = \frac{\cos a + \cos b}{1 + \cos c}$$

6+4=1

5. Define true, mean and eccentric anomalies for planet's motion. Deduce the Kepler's equation $m = \phi - e\sin\phi$, where the symbols have their usual meanings. 3+7=

(Turn Ove

Or

Describe the situation when a solar eclipse can occur. Prove that at the instant of conjunction in right ascension, the ratio of the distances of the sun from the moon and the earth is

$$\frac{\{\sin P_1 - \sin P \cos(\delta - \delta_1)\}}{\sin P_1}$$

where δ and δ_1 are the declinations of the sun and the moon, P and P_1 are their horizontal parallaxes (the square of $\sin P$ is neglected). 3+7=10

6. (a) Show that the refraction of the zenith distance of a star is $k \tan z$, where z is the apparent zenith distance and k is a constant. 5

(b) Show that the parallax in declination of a planet observed from a place in latitude ϕ vanishes if $\tan \phi = \cos H \tan \delta$, δ and H being the planet's declination and hour-angle respectively, and the earth being assumed spherical. 5

★★★

2 0 1 5

MATHEMATICS

(Major)

Paper : 5.4

(Rigid Dynamics)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : 1×7=7

   (a) Write down the moment of inertia of a rod of length $2a$ and mass M about a line through its centre perpendicular to its length.

   (b) Define equimomental bodies.

   (c) Write down the radius of gyration of a circular disc of radius a and mass M about its diameter.

   (d) State d'Alembert's principle.

   (e) Define compound pendulum.

   (f) A particle moves on a plane. What is the degree of freedom of the particle?

   (g) Define holonomic system.

2. Answer the following questions : 2×4=8

(a) Find the number of degree of freedom for a rigid body which has one point fixed but can move in space about this point.

(b) A rigid body consists of 3 particles of masses 2, 3 and 4 located at (1, −1, 1), (2, 0, 2) and (−1, 1, 0) respectively. Find the moments of inertia about x and y axes.

(c) Give a set of generalized coordinates needed to completely specify the motion of a particle constrained to move on an ellipse.

(d) If a rigid body rotates with angular velocity $\vec{\omega}$ and has angular momentum $\vec{\Omega}$, prove that the kinetic energy is given by

$$T = \frac{1}{2}\vec{\omega}\cdot\vec{\Omega}$$

3. Answer the following questions : 5×3=15

(a) Show that the moment of inertia of an ellipse of mass M and semi-axes a and b about a tangent is $\frac{5\,Mp^2}{4}$, where p is the perpendicular from the centre on the tangent.

Or

Find the moment of inertia of a uniform disc of radius a and mass M about a tangent line in its plane.

(Continued)

(b) A uniform circular disc of radius r is oscillating as a compound pendulum in a vertical plane about an axis through a point of the disc perpendicular to the plane and the length of the equivalent simple pendulum is $2r$. If h be the distance of the axis from the centre of the disc, show that

$$h : r = \sqrt{2}-1 : \sqrt{2}$$

Or

Prove in the case of a compound pendulum that the centres of suspension and oscillation are interchangeable.

(c) Obtain the Lagrange's equations for a conservative holonomic system.

Or

Use Lagrange's equations to find the differential equation for a compound pendulum which oscillates in a vertical plane about a fixed horizontal axis.

4. AB and AC are two equal similar rods freely hinged at B and lie in a straight line on a smooth table. The end A is struck by a blow perpendicular to AB, show that the resulting velocity of AB is $3\frac{1}{2}$ times that of B. 10

Or

A homogeneous sphere of radius a, rotating with angular velocity ω about a horizontal diameter, is gently placed on a table whose

(Turn Over)

coefficient of friction is μ. Show that there will be slipping at the point of contact for a time $2\omega a / 7\mu g$ and that then the sphere will roll with angular velocity $\frac{2\omega}{7}$.

5. A rod of length $2a$ is suspended by a string of length l attached to one end. If the string and rod revolve about the vertical with uniform angular velocity and their inclinations to the vertical be θ and ϕ respectively, show that

$$\frac{3l}{a} = \frac{(4\tan\theta - 3\tan\phi)\sin\phi}{(\tan\phi - \tan\theta)\sin\theta}$$ 10

Or

Show that the kinetic energy of a rigid body moving in any manner is at any instant equal to the kinetic energy of the whole mass, supposed collected at its centre of inertia and moving with it, together with the kinetic energy of the whole mass relative to its centre of inertia.

6. Answer the following questions : 5+5=10

(a) Show that the equation of the momental ellipsoid at the corner of a cube of side $2a$ referred to its principal axis is

$$2x^2 + 11(y^2 + z^2) = \text{constant}$$

(b) Show that the rotational motion of a rigid body is independent of the translatory motion of the body.

★★★

2 0 1 5

MATHEMATICS

(Major)

Paper : 5.6

(Optimization Theory)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Choose the correct option/Answer the following questions : 1×7=7

(a) Which of the following sets is not convex?

(i) $\{(x_1, x_2) : x_2 - 3 \geq x_1^2, x_1 \geq 0, x_2 \geq 0\}$

(ii) $\{(x_1, x_2) : x_1^2 + x_2^2 \leq 4\}$

(iii) $\{(x_1, x_2) : x_1 x_2 \leq 1, x_1 \geq 0, x_2 \geq 0\}$

(iv) $\{(x_1, x_2) : |x_1| \leq 1, |x_2| \leq 1\}$

(b) Given a system of m simultaneous linear equations in n unknowns $(m < n)$, the number of basic variables will be

(i) m

(ii) n

(iii) $n - m$

(iv) $n + m$

 (Turn Over)

(c) A necessary and sufficient condition for a basic feasible solution to a minimization LPP to be an optimum is that (for all j)

(i) $z_j - c_j \geq 0$

(ii) $z_j - c_j \leq 0$

(iii) $z_j - c_j = 0$

(iv) $z_j - c_j > 0$ or $z_j - c_j < 0$

(d) Express the vector $\hat{x} = (5, 9)$ as the linear combination of the vectors

$$\overline{\alpha} = (1, 2), \quad \overline{\beta} = (3, 5)$$

(e) Consider the following problems :

Max $Z_x = \overline{c}'\overline{x}$, such that $A\overline{x} \leq \overline{b}$, $\overline{x} \geq 0$ ... (A)

Min $Z_w = \overline{b}'\overline{w}$, such that $A'\overline{w} \geq c$, $\overline{w} \geq 0$ ... (B)

Fill in the blank :

If the ith constraint in A is an equality, then the ith dual variable is ______.

(f) In a balanced transportation problem with m sources and n destinations, the number of basic variables is

(i) $m+n$

(ii) $m+n+1$

(iii) $m+n-1$

(iv) None of the above

(g) In a balanced transportation problem with m sources and n destinations, the number of dual variables will be

(i) $m+n$

(ii) $m+n+1$

(iii) $m+n-1$

(iv) $m-n-1$

2. Answer the following questions : 2×4=8

(a) Is the solution

$$x_1 = 1, \ x_2 = \frac{1}{2}, \ x_3 = x_4 = x_5 = 0$$

a basic solution of the equations

$$x_1 + 2x_2 + x_3 + x_4 = 2 \text{ and}$$

$$x_1 + 2x_2 + \frac{1}{2}x_3 + x_5 = 2?$$

Justify.

(b) Can the system

$$x_1 + x_2 = 1, \ x_1 - x_3 = 2, \ x_1, \ x_2, \ x_3 \geq 0$$

have a feasible solution? Justify.

(c) Explain the term 'artificial variable' in linear programming.

(d) Show with the help of a counter example that the union of two convex sets may not be a convex set.

(Turn Over)

3. Answer any *three* of the following : 5×3=15

(a) A company sells two different products A and B. The company makes a profit of ₹ 40 and ₹ 30 per unit on products A and B respectively. The two products are produced in a common production process and are sold in two different markets. The production process has a capacity of 30000 man-hours. It takes 3 hours to produce 1 unit of A and 1 hour to produce 1 unit of B. The market has been surveyed and company officials feel that the maximum number of units of A that can be sold is 8000 units and the maximum B is 12000 units. Subject to these limitations, the products can be sold in any convex combination. Formulate an LPP and solve it graphically. 5

(b) Show that a hyperplane is a closed convex set. 5

(c) Find the dual of the following system : 5

$$\text{Min } Z_p = x_1 + x_2 + x_3$$

subject to

$$x_1 - 3x_2 + 4x_3 = 5$$
$$2x_1 - 2x_2 \le 3$$
$$2x_2 - x_3 \ge 5$$
$$x_1, x_2, x_3 \ge 0$$

(d) Show that the dual of a dual is the primal problem itself. 5

(e) What is a balanced transportation problem? Describe a transportation table. Write the names of three common methods to obtain an initial basic feasible solution for a transportation problem. 1+1+3=5

4. If A is a finite subset of E^n, then show that the convex hull of A is the set of all convex combinations of vectors of A. 10

Or

Solve the following LPP by simplex method : 10

$$\text{Min } Z = x_1 - 2x_2 + x_3$$

subject to

$$x_1 + 2x_2 - 2x_3 \le 4$$
$$x_1 - x_3 \le 3$$
$$2x_1 - x_2 + 2x_3 \le 2$$
$$x_1, x_2, x_3 \ge 0$$

5. Solve the following LPP by using the big M-method : 10

$$\text{Max } Z = -x_1 + 3x_2$$

subject to

$$x_1 + 2x_2 \ge 2$$
$$3x_1 + x_2 \le 3$$
$$x_1 \le 4$$
$$x_1, x_2 \ge 0$$

Or

Use duality to solve the following : 10

$$\text{Min } Z = 3x_1 + x_2$$

subject to

$$2x_1 + 3x_2 \geq 2$$
$$x_1 + x_2 \geq 1$$
$$x_1, x_2 \geq 0$$

6. The following table gives the cost of transporting material from supply points A, B, C, D to demand points E, F, G, H, I :

From \ To →	E	F	G	H	I
A	8	10	12	17	15
B	15	13	18	11	9
C	14	20	6	10	13
D	13	19	7	5	12

The present allocation is as follows :

A to E 90 ; A to F 10 ; B to F 150 ; C to F 10 ; C to G 50 ; C to I 120 ; D to H 210 ; D to I 70.

Check if this allocation is optimum. If not, find an optimum schedule. 5+5=10

Or

A company has 5 jobs to be done. The following matrix shows the return in ₹ of assigning ith $(i = 1, \ldots, 5)$ machine to the job $j (j = 1, \ldots, 5)$. Assign the 5 jobs to the 5 machines so as to maximize the total return : 10

Machine ↓ \ Job →	J_1	J_2	J_3	J_4	J_5
M_1	5	11	10	12	4
M_2	2	4	6	3	5
M_3	3	12	5	14	6
M_4	6	14	4	11	7
M_5	7	9	8	12	5

★★★