Total number of printed pages–8

3 (Sem–4/CBCS) MAT HC 1

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–4016

(Multivariate Calculus)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any ten*** : 1×10=10

(i) Find the domain of $f(x, y) = \dfrac{1}{\sqrt{x-y}}$.

(ii) How is directional derivative of a function at a point related to the gradient of the function at that point?

(iii) Define harmonic function?

(iv) Define $\iint\limits_R f(x, y)\, dA$.

Contd.

(v) Write the value of $\vec{\nabla}(f^n)$.

(vi) Define critical point.

(vii) Define relative extrema for a function of two variables.

(viii) When is a curve said to be positively oriented?

(ix) Describe the fundamental theorem of line integral.

(x) When is a surface said to be smooth?

(xi) Compute $\int_1^4 \int_{-2}^3 \int_2^5 dx\,dy\,dz$.

(xii) Evaluate $\underset{(x,y)\to(1,3)}{Lt} \frac{x-y}{x+y}$.

(xiii) If $f(x, y) = x^3y + x^2y^2$, find f_x.

(xiv) When is a line integral said to be path independent?

(xv) Explain the difference between $\int_C f\,ds$ and $\int_C f\,dx$.

2. Answer **any five** questions : 2×5=10

(a) Sketch the level surface $f(x, y, z) = c$ if $(x, y, z) = y^2 + z^2$ for $c = 1$.

(b) Determine f_x and f_y for $f(x, y) = xy^2 \ln(x+y)$.

(c) Find $\frac{\partial w}{\partial t}$ if $w = \ln\left(x + 2y - z^2\right)$ and $x = 2t - 1,\ y = \frac{1}{t},\ z = \sqrt{t}$.

(d) Evaluate $\int_1^2 \int_0^\pi x \cos y\,dy\,dx$.

(e) Evaluate $\underset{(x,y)\to(0,0)}{Lim} \frac{x^2 + x - xy - y}{x - y}$.

(f) Define line integral over a smooth curve.

(g) Find the Jacobian $\frac{\partial(x, y)}{\partial(u, v)}$ when $x = u + 2v,\ y = 3u - 4v$.

(h) Using polar coordinates find the limit $\underset{(x,y)\to(0,0)}{Lim} \frac{\tan\left(x^2 + y^2\right)}{x^2 + y^2}$.

3. Answer ***any four*** : 5×4=20

(a) Describe the graph of the function $f(x, y) = 1 - x - \frac{1}{2}y$.

(b) Use the method of Lagrange's multipliers to find the maximum and minimum values of $f(x, y) = 1 - x^2 - y^2$ subject to the constraints $x + y = 1$ with $x \geq 0$, $y \geq 0$.

(c) Evaluate $\int_C [(y-x)dx + x^2 y\, dy]$, where C is the curve defined by $y^2 = x^3$ from $(1,-1)$ to $(1,1)$.

(d) Examine the continuity of the following function at the origin :

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

(e) Find $\frac{\partial w}{\partial s}$ if $w = 4x + y^2 + z^3$ where $x = e^{rs^2}$, $y = ln\frac{r+s}{t}$ and $z = rst^2$.

(f) Suppose the function f is differentiable at the point P_0 and that the gradient at P_0 satisfies $\Delta f_0 \neq 0$. Show that Δf_0 is orthogonal to the level surface of f through P_0.

(g) Compute $\iint_D \left(\frac{x-y}{x+y}\right)^4 dy\,dx$ where D is the triangular region bounded by the line $x + y = 1$ and the coordinate axes, using change of variables $u = x - y$, $v = x + y$.

(h) Find the absolute extrema of $f(x, y) = 2x^2 - y^2$ on the closed bounded set S, where S is the disk $x^2 + y^2 \leq 1$.

4. Answer ***any four*** questions : 10×4=40

(a) The radius and height of a right circular cone are measured with errors of at most 3% and 2% respectively. Use increments to approximate the maximum possible percentage error in computing the volume of the cone using these measurements and the formula $V = \frac{1}{3}\pi R^2 H$.

(b) Let $f(x, y) = \begin{cases} xy\left(\dfrac{x^2 - y^2}{x^2 + y^2}\right), & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

Show that $f_x(0, y) = -y$ and $f_x(x, 0) = x$ for all x and y. Then show that $f_{xy}(0, 0) = -1$ and $f_{yx}(0, 0) = 1$.

(c) (i) Find the directional derivative of $f(x, y) = \ln(x^2 + y^3)$ at $P_0(1, -3)$ in the direction of $\vec{v} = 2i - 3j$.

(ii) In what direction is the function defined by $f(x, y) = xe^{2y-x}$ increasing most rapidly at the point $P_0(2, 1)$, and what is the maximum rate of increase? In what direction is f decreasing most rapidly?

(d) When two resistances R_1 and R_2 are connected in parallel, the total resistance R satisfies $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If R_1 is measured as 300 *ohms* with maximum error of 2% and R_2 is measured as 500 *ohms* with a maximum error of 3%, what is the maximum percentage error in R?

(e) Verify the vector field $\vec{F} = (e^x \sin y - y)i + (e^x \cos y - x - 2)j$ is conservative. Also find the scalar potential function f for $\vec{F}$.

(f) (i) Evaluate $\iiint_D \frac{dxdydz}{\sqrt{x^2 + y^2 + z^2}}$ where D is the solid sphere $x^2 + y^2 + z^2 \leq 3$.

(ii) Find the volume of the solid D, where D is bounded by the paraboloid $z = 1 - 4(x^2 + y^2)$ the xy-plane.

(g) (i) Use a polar double integral to show that a sphere of radius a has volume $\frac{4}{3}\pi a^3$.

(ii) Evaluate $\int_0^3 \int_0^{\sqrt{9-x^2}} xdydx$ by converting to polar coordinates.

(h) State Green's theorem. Verify Green's theorem for the line integral $\oint_C (y^2 dx + x^2 dy)$ where C is the square having vertices (0, 0), (1, 0), (1, 1) and (0, 1).

(i) State Stokes' theorem. Using Stokes' theorem evaluate the line integral $\oint_C \left(x^3y^2dx + dy + z^2dz\right)$, where C is the circle $x^2 + y^2 = 1$ and in the plane $z = 1$, counterclockwise when viewed from the origin.

(j) A container in R^3 has the shape of the cube given by $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$. A plate is placed in the container in such a way that it occupies that portion of the plane $x + y + z = 1$ that lies in the cubical container. If the container is heated so that the temperature at each point (x, y, z) is given by $T(x, y, z) = 4 - 2x^2 - y^2 - z^2$ in hundreds of degrees Celsius, what are the hottest and coldest points on the plate? You may assume these extreme temperatures exist.

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3 (Sem–4/CBCS) MAT HC 2

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–4026

(Numerical Methods)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any seven*** questions : 1×7=7

(a) What do you mean by an algorithm ?

(b) What is the underlying theorem of bisection method ?

(c) Write the iterative formula of secant method for solving an equation $f(x)=0$.

(d) Consider the system of equations $Ax=b$. In which method, the matrix A can be decomposed into the product of two triangular matrices ?

Contd.

(e) Name one iterative method for solving a system of linear equations.

(f) Write the iterative formula of Newton-Raphson method to find the square root of 15.

(g) What do you mean by interpolating polynomial ?

(h) Show that $\Delta = E - 1$.

(i) What do you mean by numerical differentiation ?

(j) Write the formula for second order central difference approximation to the first derivative.

2. Answer ***any four*** questions : 2×4=8

(a) Examine whether the fixed point iteration method is applicable for finding the root of the equation :

$$2x = \sin x + 5.$$

(b) Define rate of convergence and order of convergence of a sequence.

(c) Prove that $\mu = \left(1 + \frac{\delta^2}{4}\right)^{1/2}$ where μ and δ are average and central difference operators.

(d) Verify that the following equation has a root on the interval $(0,1)$:

$$f(x) = \ln(1+x) - \cos x = 0.$$

(e) If $P_1(x) = a_0 + a_1 x$ such that $P_1(x_0) = f_0$ and $P_1(x_1) = f_1$, then obtain an expression for $P_1(x)$ in terms of x_i's and f_i's $(i = 0, 1)$.

(f) Show that $\delta = \nabla(1 - \nabla)^{-\frac{1}{2}}$.

(g) What do you mean by degree of precision of a quadrature rule ? If a quadrature rule $I_n(f)$ integrates $1, x, x^2$ and x^3 exactly, but fails to integrate x^4 exactly, then what will be the degree of precision of $I_n(f)$?

(h) Mention briefly about the use of Euler's method.

3. Answer ***any three*** questions : 5×3=15

(a) Give a brief sketch of the method of false position.

(b) Give the geometrical interpretation of Newton-Raphson method.

(c) Construct an algorithm for the secant method.

(d) Show that an LU decomposition is unique up to scaling by a diagonal matrix.

(e) Discuss about the advantages and disadvantages of Lagrange's form of interpolating polynomial.

(f) Given $f(2)=4$, $f(2.5)=5.5$, find the linear interpolating polynomial using Lagrange's interpolation. Hence find an approximate value of $f(2.2)$.

(g) Derive the closed Newton-Cotes quadrature formula corresponding to $n=1$. Why is this formula called trapezoidal rule ?

(h) Evaluate $\int_0^1 tan^{-1} x\, dx$ using Simpson's $\frac{1}{3}$rd rule.

4. Answer ***any three*** questions : 10×3=30

(a) Perform five iterations of the bisection method to obtain the smallest positive root of the equation :

$$f(x)=x^3-5x+1=0.$$

(b) Apply Newton-Raphson method to determine a root of the equation :

$$f(x)=cos\,x-xe^x=0.$$

Taking the initial approximation as $x_0=1$, perform five iterations.

(c) Form an LU decomposition of the following matrix :

$$A=\begin{pmatrix}1 & 4 & 3\\ 2 & 7 & 9\\ 5 & 8 & -2\end{pmatrix}$$

(d) Find the order of convergence of the iterative method $x_{n+1}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right)$ to compute an approximation to the square root of a positive real number a. To find the real root of $x^3-x-1=0$ near $x=1$, which of the following iteration functions give convergent sequences ?

(i) $x=x^3-1$

(ii) $x=\frac{x+1}{x^2}$

(e) Construct the difference table for the sequence of values :

$$f(x) = (0, 0, 0, \varepsilon, 0, 0, 0).$$

where ε is an error. Also show that —

(i) the error spreads and increases in magnitude as the order of differences is increased;

(ii) the errors in each column have binomial coefficients.

(f) Let $x_0 = -3$, $x_1 = 0$, $x_2 = e$ and $x_3 = \Pi$. Determine formulas for the Lagrange's polynomials $L_{3,0}(x)$, $L_{3,1}(x)$, $L_{3,2}(x)$ and $L_{3,3}(x)$ associated with the given interpolating points.

(g) For the function $f(x) = ln\, x$, approximate $f'(3)$ using —

(i) first order forward difference, and

(ii) first order backward difference approximation formulas.

[Starting with step size $h = 1$, reduce it by $\frac{1}{10}$ in each step until convergence.]

5+5=10

(h) Solve the initial value problem :

$$\frac{dx}{dt} = 1 + \frac{x}{t}, \quad 1 \le t \le 2.5$$

$$x(1) = 1,$$

using Euler's method with step size $h = 0.5$ and find an approximate value of $x(2.5)$.

Total number of printed pages–11

3 (Sem–4/CBCS) MAT HC 3

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–4036

(Ring Theory)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any ten*** : $1 \times 10 = 10$

   (a) The set Z of integers under ordinary addition and multiplication is a commutative ring with unity 1. What are the units of Z?

   (b) What is the trivial subring of R?

Contd.

(c) What are the elements of $Z_3[i]$?

(d) Give the definition of zero divisor.

(e) Give an example of a commutative ring without zero divisors that is not an integral domain.

(f) What is the characteristic of an integral domain ?

(g) Why is the idea $\langle x^2+1\rangle$ not prime in $Z_2[x]$?

(h) Find all maximal ideals in Z_8.

(i) Is the mapping from Z_5 to Z_{30} given by $x \rightarrow 6x$ is a ring homomorphism ?

(j) If ϕ is an isomorphism from a ring R onto a ring S, then ϕ^{-1} is an isomorphism from S onto R.
Write True *or* False.

(k) Is the ring $2z$ isomorphic to the ring $3z$?

(l) Let $f(x)=x^3+2x+4$ and $g(x)=3x+2$ is $z_5[x]$. Determine the quotient and remainder upon dividing $f(x)$ by $g(x)$.

(m) Why is the polynomial

$$3x^5+15x^4-20x^3+10x+20$$

irreducible over Q ?

(n) Give the definition of Euclidean domain.

(o) State the second isomorphism theorem for rings.

2. Answer ***any five*** : 2×5=10

(a) Define ring. What is the unity of a polynomial ring $Z[x]$?

(b) Prove that in a ring R, $(-a)(-b) = ab$ for all $a, b \in R$.

(c) Prove that set S of all matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ with a and b, forms a sub-ring of the ring R of all 2×2 matrices having elements as integers.

(d) Let R be a ring with unity 1. If 1 has infinite order under addition, then the characteristic of R is 0. If 1 has order n under addition, then prove that the characteristic of R is n.

(e) Let

$z/4z = \{0+4z, 1+4z, 2+4z, 3+4z\}$.

Find $(2+4z)+(3+4z)$ and $(2+4z)(3+4z)$.

(f) Let $R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \middle| a, b \in Z \right\}$ and let ϕ be the mapping defined as $\begin{bmatrix} a & b \\ b & a \end{bmatrix} \to a - b$.

Show that ϕ is a homomorphism.

(g) Let $f(x) = 4x^3 + 2x^2 + x + 3$ and

$g(x) = 3x^4 + 3x^3 + 3x^2 + x + 4$

where $f(x), g(x) \in Z_5[x]$.

Compute $f(x) + g(x)$ and $f(x) \cdot g(x)$.

(h) Prove that in an integral domain, every prime is an irreducible.

3. Answer ***any four*** : 5×4=20

(a) Define a sub-ring. Prove that a non-empty subset S of a ring R is a sub-ring if S is closed under subtraction and multiplication, that is if $a-b$ and ab are in S whenever a and b are in S. 1+4=5

(b) Prove that the ring of Gaussian integers $Z[i]=[a+ib \mid a, b \in Z]$ is an integral domain.

(c) Let R be a commutative ring with unity and let A be an ideal of R. Then prove that R/A is an integral domain if and only if A is prime.

(d) If D is an integral domain, then prove that $D[x]$ is an integral domain.

(e) (i) If R is commutative ring then prove that $\phi(R)$ is commutative, where ϕ is an isomorphism on R. 3

(ii) If the ring R has a unity 1, $S \neq \{0\}$ and $\phi: R \to S$ is onto, then prove that $\phi(1)$ is the unity of S. 2

(f) Let $f(x) \in Z[x]$. If $f(x)$ is reducible over Q, then prove that it is reducible over Z.

(g) Consider the ring

$S=\left\{\begin{bmatrix} a & b \\ -b & a \end{bmatrix} a, b \in Z\right\}$. Show that

$\phi: \mathbb{C} \to S$ is given by

$\phi(a+bi)=\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ is a ring isomorphism.

(h) Prove that $Z[i]=\{a+bi \mid a, b \in Z\}$, the ring of Gaussian integers is an Euclidean domain.

4. Answer **any four** : 10×4=40

(a) (i) Prove that the set of all continuous real-valued functions of a real variablc whose graphs pass through the point $(1, 0)$ is a commutative ring without unity under the operation of pointwise addition and multiplication [that is, the operations $(f+g)(a)=f(a)+g(a)$ and $(f.g)(a)=f(a)\cdot g(a)$. 6

Contd.

(ii) Prove that if a ring has a unity, it is unique and if a ring element has an inverse, it is unique. 4

(b) Define a field. Is the set I of all integers a field with respect to ordinary addition and multiplication? Let $Q[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in Q$. Prove that $Q[\sqrt{2}]$ is a field. 2+1+7=10

(c) (i) Prove that the intersection of any collection of subrings of a ring R is a sub-ring of R. 5

(ii) Let R be a commutative ring with unity and let A be an ideal of R. Prove that R/A is a field if A is maximal. 5

(d) Define factor ring. Let R be a ring and let A be a subring of R. Prove that the set of co-sets $\{r + A \mid r \in R\}$ is a ring under the operation

$(s+A)+(t+A)=(s+t)+A$ and

$(s+A)(t+A)=st+A$ if and only if A is an ideal of R. 1+5+4=10

(e) (i) Let ϕ be a ring homomorphism from R to S. Prove that the mapping from $R/\ker\phi$ to $\phi(R)$, given by $r+\ker\phi \rightarrow \phi(r)$ is an isomorphism. 5

(ii) Let R be a ring with unity and the characteristic of R is $n > 0$. Prove that R contains a sub-ring isomorphic to Z_n. If the characteristic of R is 0, then prove that R contains a sub-ring isomorphic to Z. 3+2=5

(f) Let F be a field and let $p(x) \in F[x]$. Prove that $\langle p(x) \rangle$ is a maximal ideal in $F[x]$ if and only if $p(x)$ is irreducible over F.

(g) Let F be a field and let $f(x)$ and $g(x) \in F[x]$ with $g(x) \neq 0$. Prove that there exists unique polynomials $q(x)$ and $r(x)$ in $F[x]$ such that $f(x) = g(x)\,q(x) + r(x)$ and either $r(x) = 0$ or $deg\,r(x) \langle deg\,g(x)$. With the help of an example verify the division algorithm for $F[x]$. 7+3=10

(h) (i) If F is a field, then prove that $F[x]$ is a principal ideal domain. 5

(ii) Let F be a field and let $p(x)$, $a(x)$, $b(x) \in F[x]$. If $p(x)$ is irreducible over F and $p(x) \mid a(x)\,b(x)$, then prove that $p(x) \mid a(x)$ or $p(x) \mid b(x)$. 5

(i) Prove that every principal ideal domain is a unique factorization domain.

(j) (i) Prove that every Euclidean domain is a principal ideal domain. 5

(ii) Show that the ring

$$Z[\sqrt{-5}] = \{a + b\sqrt{-5} \mid ab \in Z\}$$

is an integral domain but not a unique factorization domain. 5
