Total number of printed pages–11

3 (Sem–1/CBCS) MAT HC 1

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–1016

(Calculus)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any seven*** of the following questions : $1 \times 7 = 7$

   (a) Write down the n^{th} derivative of $\cos(5x+3)$.

   (b) Write when the graph of a function f is said to have vertical tangent at a point $P(c, f(c))$.

Contd.

(c) Write down the value of $\underset{x \to +\infty}{Lt}\ x^n e^{-kx}$

(d) Evaluate $\int_0^{\pi/2} \sin^6 x\, dx$

(e) In terms of marginal revenue and marginal cost, when is the profit maximized?

(f) For what purpose the disk and washer methods are used?

(g) Parameterize the curve $y = 4x^2$

(h) When the graph of a vector function $\vec{F}(t)$ is said to be smooth?

(i) Determine the values of t for which the vector function $\vec{F}(t) = \dfrac{\hat{i} + 2\hat{j}}{t^2 + 1}$ is continuous?

(j) State the geometrical significance of the scalar triple product of vectors $\vec{u}, \vec{v}$ and $\vec{w}$.

(k) Find $\int_0^{\pi} \left(t\hat{i} + 3\hat{j} - \sin t\,\hat{k}\right) dt$

(l) When a function f is said to be continuously differentiable on an interval I?

2. Answer **any four** of the following questions: 2×4=8

(a) Evaluate $\underset{x \to +\infty}{Lt} \dfrac{3x^3 - 5x + 9}{5x^3 + 2x^2 - 7}$

(b) Using Leibnitz's rule obtain the n^{th} derivative of $y = x^3 e^x$.

(c) By integration find the length of the circle $r = 2\sin\theta$.

(d) Let $\vec{F}(t) = \hat{i} + e^t \hat{j} + t^2 \hat{k}$ and $\vec{G}(t) = 3t^2 \hat{i} + e^{-t} \hat{j} + 2t \hat{k}$,

then find $\frac{d}{dt}\{\vec{F}(t) \cdot \vec{G}(t)\}$.

(e) Find the tangent vector to the graph of the vector function $\vec{F}(t) = t^2 \hat{i} + 2t \hat{j} + e^t \hat{k}$ at the point $t = -1$.

(f) State Kepler's laws of motion.

(g) Find the volume generated by revolving about OX, the area bounded by $y = x^3$ between $x = 0$ and $x = 2$.

(h) Find the length of the polar curve $r = e^{3\theta}, 0 \le \theta \le \pi/2$.

3. Answer ***any three*** of the following : 5×3=15

(a) Find the constants a and b that guarantee that the graph of the function defined by $f(x) = \frac{ax+5}{3-bx}$.

will have a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = 3$.

(b) Evaluate : 2+3=5

(i) $\underset{x \to \pi/2^-}{Lt} (x - \pi/2) \tan x$

(ii) $\underset{x \to +\infty}{Lt} \left(1 + \frac{1}{2x}\right)^{3x}$

(c) If $I_n = \int_0^{\pi/4} \tan^n \theta \, d\theta$, then prove that

$n(I_{n+1} + I_{n-1}) = 1$.

Hence evaluate $\int_0^{\pi/4} \tan^3 \theta \, d\theta$. 3+2=5

(d) A firm determines that x units of its product can be sold daily at p rupees per unit where $x = 1000 - p$. The cost of producing x units per day is $C(x) = 3000 + 20x$.

Find the revenue function $R(x)$.

Find the profit function $P(x)$.

Assuming that the production capacity is at most 500 units per day, determine how many units the company must produce and sell each day to maximize the profit. 1+1+3=5

(e) Show that a cone of radius r and height h has lateral surface area $S = \pi r\sqrt{r^2 + h^2}$.

(f) For any three vectors $\vec{a}, \vec{b}, \vec{c}$ in space, prove that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{c} . \vec{a})\, \vec{b} - (\vec{b} . \vec{a})\, \vec{c}$.

(g) Use cylindrical shell method to find the volume of the solid generated when the region R under $y^2 = x$ and x-axis over the interval [0, 4] is revolved about the line $y = -1$.

(h) If the non-zero vector function $\vec{F}(t)$ is differentiable and has constant length, then prove that $\vec{F}(t)$ is orthogonal to the derivative vector $\vec{F}'(t)$.

Verify this result for

$\vec{F}(t) = \cos t\, \hat{i} + \sin t\, \hat{j} + 3\hat{k}$. 3+2=5

4. Answer ***any three*** of the following : 10×3=30

(a) State Leibnitz's theorem. Use it to show that if $y = e^{m \cos^{-1} x}$, then

$$\left(1 - x^2\right) y_{n+2} - (2n+1)\, x y_{n+1} - \left(n^2 + m^2\right) y_n = 0$$

Hence find $y_n(0)$. 2+5+3=10

(b) Find the vertical and horizontal asymptotes (if any) of the graph of the function

$$f(x) = \frac{x^2 - x - 2}{x - 3}.$$

Find where the graph is rising, where it is falling, determine concavity, locate all critical points and points of inflection. Finally sketch the graph.

(c) Obtain the reduction formula for $\int \sin^n x \, dx$.

Hence evaluate

(i) $\int_0^{\pi/2} \sin^n x \, dx$

(ii) $\int_0^{\pi/2} \sin^7 x \, dx$

5+3+2=10

(d) A boy standing at the edge of a cliff throws a ball upward at an angle of 30° with an initial speed of $64 ft/s$. Suppose that when the ball leaves the boy's hand, it is $48 ft$ above the ground at the base of the cliff.

(i) What are the time of flight of the ball and its range?

(ii) What are the velocity of the ball and its speed at impact?

(iii) What is the highest point reached by the ball during its flight?

3+3+4=10

(e) (i) Find the area of the surface generated by revolving about the x-axis the top half of the cardioid $r = 1 + \cos\theta$. 5

(ii) Using disk method find the volume generated when the region bounded by the line $y = 4 - x$ and the x-axis on the interval $0 \le x \le 4$ revolve about the line $x = -2$.

5

(f) (i) Find the position vector $\vec{R}(t)$ and velocity vector $\vec{V}(t)$, given the acceleration $\vec{A}(t)$ and initial position and velocity vectors $\vec{R}(0)$ and $\vec{V}(0)$ as

$\vec{A}(t) = t^2\hat{i} - 2\sqrt{t}\,\hat{j} + e^{3t}\hat{k}$

$\vec{R}(0) = 2\hat{i} + \hat{j} - \hat{k}, \vec{V}(0) = \hat{i} - \hat{j} - 2\hat{k}$.

5

(ii) A particle moves along the parametric curve $x = 2t$, $y = t$. Find the position vector $\vec{R}(t)$ and velocity vector $\vec{V}(t)$ in terms of $\hat{U}_r$ and $\hat{U}_\theta$. 5

(g) (i) It is projected that t years from now, the population of a certain country will be $P(t) = 50e^{0.02t}$ million.

At what rate will the population be changing with respect to time 10 years from now?

At what percentage of rate, will the population be changing with respect to time t years from now? 3+3=6

(ii) Find the length of the curve defined by $9x^2 = 4y^3$ between the points $(0, 0)$ and $(2\sqrt{3}, 3)$. 4

(h) A object moving along a smooth curve has velocity $\vec{V}$ given by $\vec{V} = \frac{ds}{dt}\hat{T}$.

Deduce the expression for acceleration in the form $\bar{A} = \frac{d^2s}{dt^2}\hat{T} + k\left(\frac{ds}{dt}\right)^2 \hat{N}$

where s is the arc length along the trajectory and k is the curvature. For an object moving along a helix with position vector $\vec{R}(t) = (\cos t, \sin t, t)$ at any instant t, find the tangential and normal components of acceleration. 5+5=10

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3 (Sem-1/CBCS) MAT HC 2

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–1026

(Algebra)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any ten*** : $1\times10=10$

(a) Find the polar representation of $z=-3i$.

(b) State De Moivre's theorem.

(c) Let $z_0=r(\cos t^*+i\sin t^*)$ be a complex number with $r>0$ and $t^*\in[0, 2\pi)$. Write down the formula for n distinct n^{th} roots of z_0.

Contd.

(d) Identify the quantifier, set of context and property in the statement, "Every student in this classroom is at least 5*ft* tall."

(e) Define implication. Give an example.

(f) Prove by contradiction "There is no greatest integer".

(g) Let A and B be two sets, write when $A \times B = \phi$. Justify your answer.

(h) What is domain and range for the function $f(x) = \tan x$.

(i) What are the options about the solutions of a system of linear equations?

(j) Determine h such that the matrix

$\begin{bmatrix} 2 & 3 & h \\ 6 & 9 & 5 \end{bmatrix}$ is the augmented matrix of

a consistent linear system.

(k) State True **or** False with justification : "Whenever a system has free variables the solution set is infinite."

(l) Write down the system of equations that is equivalent to the vector equation

$$x_1\begin{bmatrix} -2 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 8 \\ 5 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ -6 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(m) Define Pivot positions in a matrix.

(n) Prove $\vec{U} + \vec{V} = \vec{V} + \vec{U}$ for any $\vec{U} . \vec{V}$ in $\mathbb{R}^n$.

(o) Write the system of equation as a matrix equation

$$\begin{aligned} 3x_1 + x_2 - 5x_3 &= 9 \\ x_2 + 4x_3 &= 0 \end{aligned}$$

(p) Given, $A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Compute $x^T A^T$ and $A^T x^T$.

(q) A is an $n \times n$ matrix. Prove statement (i) $\Rightarrow$ statement (ii).

(i) A is an invertible matrix

(ii) $\exists a$ $n \times n$ matrix C s.t. $CA = I$

(r) A is an $n \times n$ matrix

Fill in the blank :

If two rows of A are interchanged to produce B, then det $B =$ ________ .

2. Answer ***any five*** : 2×5=10

(a) If $z_1 = 1 - i$ and $z_2 = \sqrt{3} + i$. Express $z_1 z_2$ in polar form.

(b) Write the 'converse' and 'contrapositive' of the following statement :

"For real numbers x and y, if xy is an irrational number then either x is irrational or y is irrational."

(c) Why may we use the contrapositive of a statement to prove the statement instead of direct proof? Justify using truth table.

(d) Produce counter examples to disapprove the following :

(i) For $x, y \in \mathbb{R}$, $|a| > |b|$ if $a > b$

(ii) For any $x \in \mathbb{R}$, $x^2 \geq x$

(e) Express the empty set as a subset of $\mathbb{R}$ in two different ways.

(f) Express $\mathbb{N}$ as the union of an infinite number of finite sets I_n indexed by $n \in \mathbb{N}$.

(g) Give an example of a relation that is not reflexive, not transitive but is symmetric.

(h) State True **or** False with justification :

An example of a linear combination of vectors $\vec{v}_1$ and $\vec{v}_2$ is $\frac{1}{2}\vec{v}_1$.

(i) Prove that the following vectors are linearly dependent

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}.$$

(j) Evaluate the determinant by using row reduction to Echelon form

$$\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix}$$

3. Answer **any four** : 5×4=20

(a) Compute $z = (1+i\sqrt{3})^n + (1-i\sqrt{3})^n$.

(b) Prove that the power set of a set with n elements has 2^n elements. Write down the power set of $S = \{a, b\}$.

(c) Prove that the equivalence classes of an equivalence relation on a set X induces a partition of X.

(d) Prove $(1+x)^n \geq 1+nx$ for $x \in \mathbb{R}$ such that $x > -1$ and for each $n \in N$. Give the name of this inequality.

(e) Balance the chemical equation using vector equation approach the following reaction between potassium permanganate ($KMnO_4$) and manganese sulfate ($MnSO_4$) in water produces manganese dioxide, potassium sulfate and sulfuric acid.

The unbalanced equation is

$$KMnO_4 + MnSO_4 + H_2O \to MnO_2 + K_2SO_4 + H_2SO_4$$

(f) Find the value of h for which the set of vectors is linearly dependent

$$\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix}, \begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix}$$

(g) Let A be an $m \times n$ matrix. Prove that the following statements are logically equivalent.

(i) For each $b \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.

(ii) Each $b \in \mathbb{R}^m$ is a linear combination of the columns of A.

(iii) The columns of A span $\mathbb{R}^m$.

(iv) A has a pivot position in every row.

(h) Use Cramer's rule to compute the solutions to the system

$$\begin{aligned} 2x_1 + x_2 \quad &= 7 \\ -3x_1 \quad + x_3 &= -8 \\ x_2 + 2x_3 &= -3 \end{aligned}$$

4. Answer ***any four*** : 10×4=40

(a) (i) Prove $\prod_{\substack{1\le k\le n-1 \\ gcd(k,n)=1}} \sin\frac{k\pi}{n} = \frac{1}{2^{\phi(n)}}$ whenever n is not a power of a prime. 5

(ii) Solve the equation

$z^7 - 2iz^4 - iz^3 - 2 = 0$ 5

(b) For any three sets A, B and C, show that

(i) $(A \cup B)\setminus(A \cap B) = (A \setminus B) \cup (B \setminus A)$ 5

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ 5

(c) Define graph of a function verify that the set $\{(x, y) \in \mathbb{R}^2 : x = |y|\}$ is not the graph of any function. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax^2 + bx + c, a \neq 0$. Show that the function is neither one-one nor onto. 2+2+6=10

(d) Let $X = \mathbb{R}$ and let $R = \{(x, y) \in R^2 : xy = 0\}$. When $x \in \mathbb{R}$ is related to $y \in \mathbb{R}$? Define reflexive, symmetric, antisymmetric and transitive relation with examples. 2+2+2+2+2=10

(e) If $A \subseteq N$, what is the least element of A? State and prove Division Algorithm. 2+1+7=10

(f) (i) Solve the system : 5

$$x_1 - 3x_2 + 4x_3 = -4$$
$$3x_1 - 7x_2 + 7x_3 = -8$$
$$-4x_1 + 6x_2 - x_3 = 7$$

(ii) Suppose the system 3

$$x_1 + 3x_2 = f$$
$$cx_1 + dx_2 = g$$

is consistent for all possible values of f and g, what can you say about the co-efficients c and d. Justify.

(iii) Suppose a 3 × 5 co-efficient matrix for a system has three pivot columns. Is the system consistent? Justify. 2

(g) (i) If $\vec{U} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\vec{V} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

Display $\vec{U}, \vec{V}, \vec{U} - \vec{V}$ using arrows on an xy graph. 3

(ii) List five vectors in the span $\{\vec{v}_1, \vec{v}_2\}$

$$\vec{v}_1 = \begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$$ 2

(iii) Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Does $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span $\mathbb{R}^4$? Justify. 5

(h) (i) Describe all solutions of $A\vec{x} = \vec{0}$ in parametric vector form

$$A = \begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ 5

(ii) Does $A\vec{x} = \vec{b}$ have at least one solution for every possible $\vec{b}$ if A is a 3×2 matrix with two pivot positions? 2

(iii) Prove that if a set contains more vectors than the number of entries in each vector then the set is linearly dependent. 3

(i) (i) Define linear transformation. Give an example. 2

(ii) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then prove T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$. 3

(iii) Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which is a horizontal shear transformation that leaves e_1 unchanged and maps e_2 into $e_2 + 3e_1$. 3

(iv) Show that T is a linear transformation by finding a matrix that implements the mapping

$$T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3,\ x_2 - 6x_3)$$ 2

(j) (i) Find the inverse of the matrix A (if it exists) by performing suitable row operations on the augmented matrix $[A : I]$ where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}.$$ 4

(ii) Find the volume of the parallelopiped with one vertex at the origin and adjacent vertices at $(1, 0, -2)$, $(1, 2, 4)$ and $(7, 1, 0)$. 3

(iii) Let the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ be determined by a 2×2 matrix A. Prove that if S is a parallelogram in $\mathbb{R}^2$ then $\{\text{area of } T(S)\} = /\det A/ \{\text{area of } S\}$ 3
