Total number of printed pages–11

3 (Sem–3 /CBCS) MAT HC 1

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–3016

(Theory of Real Functions)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any ten*** parts : 1×10=10

(a) Is every point in I a limit point of $I \cap Q$?

(b) Find $\lim_{x \to 1} \frac{x^2 - x + 1}{x + 1}$.

(c) Let $f(x) = sgn(x)$. Write the limits $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$.

(d) Let $p:\mathbb{R}\to\mathbb{R}$ be the polynomial function

$$p(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

if $a_n > 0$, then $\lim_{x\to\infty} p(x) = ?$

(e) Let f be defined on $(0, \infty)$ to $\mathbb{R}$. Then the statement

" $\lim_{x\to\infty} f(x) = L$ if and only if

$\lim_{x\to 0^+} f\left(\frac{1}{x}\right) = 1$ " is true *or* false.

(f) Let $A \subseteq \mathbb{R}$ and let $f_1, f_2, \ldots, f_n$ be function on A to $\mathbb{R}$, and let c be a cluster point of A. If $\lim_{x\to c} f_k(x) = L_k$,

$k = 1, 2, \ldots, n$,

then $\lim_{x\to c} (f_1 \cdot f_2 \cdot \ldots \cdot f_n) = ?$

(g) Is the function $f(x) = \dfrac{1}{x}$ continuous on $A = \{x \in \mathbb{R} : x > 0\}$?

(h) Write the points of continuity of the function $f(x) = |x|$.

(i) "A rational function is continuous at every real number for which it is defined." Is it true *or* false ?

(j) "Let f, g be defined on $\mathbb{R}$ and let $c \in \mathbb{R}$. If $\lim_{x \to c} f(x) = b$ and g is continuous at b, then $\lim_{x \to c} (g \cdot f)(x) = g(b)$." Write whether this statement is correct or not.

(k) The functions $f(x) = x$ and $g(x) = \sin x$ are uniformly continuous on $\mathbb{R}$. Is fg uniformly continuous on $\mathbb{R}$? If not, give the reason.

(l) A continuous periodic function on R is bounded and ______ on $\mathbb{R}$.

(Fill in the blank)

(m) "The derivative of an odd function is an even function." Write true ***or*** false.

(n) Write the derivative of the function $f(x) = |x|$ for $x \neq 0$.

(o) If f is differentiable on $[a, b]$ and g is a function defined on $[a, b]$ such that $g(x) = kx - f(x)$ for $x \in [a, b]$. If $f'(a) < k < f'(b)$, then find $g'(c)$.

(p) "Suppose $f : [0, 2] \rightarrow \mathbb{R}$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$, with $f(0) = 0$, $f(2) = 1$. If there exists $c \in (0, 2)$, then $f'(c) = \frac{1}{3}$." Is it true *or* false?

(q) Find $\lim_{x \to 0} \frac{x^2 + x}{\sin 2x}$.

(r) "The function $f(x) = 8x^3 - 8x^2 + 1$ has two roots in $[0, 1]$." Write true ***or*** false.

2. Answer ***any five*** parts : 2×5=10

(a) Use the definition of limit to show that

$$\lim_{x \to 2} \left(x^2 + 4x\right) = 12.$$

(b) Find $\lim_{x \to 0} x \sin\left(\frac{1}{x^2}\right)$, $(x \neq 0)$.

(c) Give an example of a function that has a right-hand limit but not a left-hand limit at a point.

(d) Define $g:\mathbb{R}\to\mathbb{R}$ by

$$g(x)=\begin{cases}2x & \text{for } x\in Q\\ x+3, & \text{for } x\in Q^c\end{cases}$$

Find all points at which g is continuous.

(e) Show that the '*sine*' function is continuous on $\mathbb{R}$.

(f) Show that the function $f(x)=\frac{1}{x}$ is uniformly continuous on $[a,\infty]$, where $a>0$.

(g) Using the mean value theorem, show that

$$\frac{x-1}{x}<ln\,(x)<x-1 \text{ for } x>1.$$

(h) Show that $f(x)=x^{1/3}$, $x\in\mathbb{R}$, is not differentiable at $x=0$.

(i) Let $f(x)=\frac{ln(\sin x)}{ln(x)}$

Find $\lim_{x\to 0^+} f(x)$.

(j) State Darboux's theorem.

3. Answer ***any four*** parts : 5×4=20

(a) Prove that a number $c \in \mathbb{R}$ is a cluster point of a subset A of $\mathbb{R}$ if and only if there exists a sequence (x_n) in A such that $\lim_{n\to\infty} x_n = c$ and $x_n \neq c$ for all $n \in \mathbb{N}$.

(b) State and prove squeeze theorem.

(c) Let $A \subseteq \mathbb{R}$, let f and g be functions on A to $\mathbb{R}$, and let f and g be continuous at a point c in $\mathbb{A}$. Prove that $f-g$ and fg are continuous at c.

(d) Give an example of functions f and g that are both discontinuous at a point c in $\mathbb{R}$ such that $f+g$ and fg are continuous at c.

(e) If $f : A \to \mathbb{R}$ is a Lipschitz function, then prove that f is uniformly continuous on A.

(f) Determine where the function

$$f(x) = |x| + |x-1|$$

from $\mathbb{R}$ to $\mathbb{R}$ is differentiable and find the derivative.

(g) Find $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$.

(h) Determine whether *or* not $x = 0$ is a point of relative extremum of the function $f(x) = x^3 + 2$.

4. Answer ***any four*** parts : 10×4=40

(a) Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. Prove that the following are equivalent :

(i) $\lim_{x \to c} f(x) = L$

(ii) Given any ε-neighbourhood $V_\varepsilon(L)$ of L, there exists a δ-neighbourhood $V_\delta(c)$ of c such that if $x \neq c$ is any point $V_\delta(c) \cap A$, then $f(x)$ belongs to $V_\varepsilon(L)$.

(b) (i) Find $\lim\limits_{x \to 0} \dfrac{\sqrt{1+2x} - \sqrt{1+3x}}{x+2x^2}$, where $x > 0$. 4

(ii) Prove that $\lim\limits_{x \to 0} \cos\left(\frac{1}{x}\right)$ does not exist but $\lim\limits_{x \to 0} x \cos\left(\frac{1}{x}\right) = 0$. 6

(c) (i) Let $f(x) = e^{\frac{1}{x}}$ for $x \neq 0$. Show that $\lim\limits_{x \to 0^+} f(x)$ does not exist in $\mathbb{R}$ but $\lim\limits_{x \to 0^-} f(x) = 0$. 5

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x+y) = f(x) + f(y)$ for all x, y in $\mathbb{R}$. Suppose that $\lim\limits_{x \to 0} f(x) = L$ exists. Show that $L = 0$ and then prove that f has a limit at every point c in $\mathbb{R}$. 5

(d) (i) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that f is not continuous at any point of $\mathbb{R}$. 5

(ii) Prove that every polynomial function is continuous on $\mathbb{R}$. 5

(e) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $|f|$ be defined by $|f|(x) = |f(x)|$ for $x \in A$. Also let $f(x) \geq 0$ for all $x \in A$ and let $\sqrt{f}$ be defined by $(\sqrt{f})(x) = \sqrt{f(x)}$ for $x \in A$. Prove that if f is continuous at a point c in A, then $|f|$ and $\sqrt{f}$ are continuous at c. 5+5=10

(f) (i) State and prove Bolzano's intermediate value theorem. 1+4=5

(ii) Let A be a closed bounded interval and let $f : A \to \mathbb{R}$ is continuous on A. Prove that f is uniformly continuous on A. 5

(g) Let $A \subseteq \mathbb{R}$ be an interval, let $c \in A$, and let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions differentiable at c. Prove that

(i) the function $f+g$ is differentiable at c and

$$(f+g)'(c) = f'(c) + g'(c)$$ 5

(ii) if $g(c) \neq 0$, then the function f/g is differentiable at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)\,g(c) - f(c)\,g'(c)}{(g(c))^2}$$ 5

(h) State and prove Rolle's theorem. Give the geometrical interpretation of the theorem. (2+5)+3=10

(i) (i) Use Taylor's theorem with $n = 2$ to approximate $\sqrt[3]{1+x},\ x > -1$. 5

(ii) If $f(x) = e^x$, show that the remainder term in Taylor's theorem converges to zero as $n \to \infty$ for each fixed x_0 and x. 5

(j) Find the limits : 5+5=10

(i) $\lim_{x \to 0^+} x^{\sin x}$

(ii) $\lim_{x \to \frac{\pi}{2}^-} \frac{\tan x}{\sec x}$

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3 (Sem–3/CBCS) MAT HC 2

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–3026

(Group Theory–I)

Full Marks : 80

Time : Three hours.

The figures in the margin indicate full marks for the questions.

1. Answer ***any ten*** questions : 1×10=10

(a) What do you mean by the symmetry group of a plane figure ?

(b) The set S of positive irrational numbers together with 1 is a group under multiplication. Justify whether it is true ***or*** false.

Contd.

(c) Define a binary operation on the set $\{0, 1, 2, 3, 4, 5\}$ for which it is a group.

(d) Let $G = \langle a \rangle$ be a cyclic group of order n. Write a necessary and sufficient condition for which a^k is a generator of G.

(e) What do you mean by even permutation? Give an example.

(f) Write the order of the alternating group of degree n.

(g) Let $G = S_3$ and $H = \{(1), (13)\}$. Write the left cosets of H in G.

(h) Show that there is no isomorphism from Q, the group of rational numbers under addition, to $Q^{\#}$, the group of non-zero rational numbers under multiplication.

(i) State Cayley's theorem.

(j) Let $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ be defined by $\phi(x) = 3x$, $x \in \mathbb{Z}_{12}$. Find ker ϕ.

(k) On the set $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$, define a binary operation for which it is a group.

(l) Define normalizer of an element in a group G.

(m) Product of two subgroups of a group is again a subgroup. State whether true **or** false.

(n) State Lagrange's theorem.

(o) What is meant by external direct product of a finite number of groups?

(p) Find the order of the permutation

$$f = \begin{pmatrix} a & b & c & d & e \\ c & a & b & e & d \end{pmatrix}$$

(q) The subgroup of an abelian group is abelian. State whether it is true **or** false.

(r) Give the statement of third isomorphism theorem.

2. Answer ***any five*** questions : 2×5=10

(a) Show that in a group G, right and left cancellation laws hold.

(b) Show that a group of prime order is cyclic.

(c) Every subgroup of an abelian group is normal. Justify whether it is true ***or*** false.

(d) Let $\mathbb{C}^*$ denote the group of non-zero complex numbers under multiplication. Define $\phi : \mathbb{C}^* \to \mathbb{C}^*$ by $\phi(x) = x^4, x \in \mathbb{C}^*$. Show that ϕ is a homomorphism and find $\ker \phi$.

(e) If ϕ is an isomorphism from a group G onto a group $\overline{G}$, then show that ϕ carries the identity element of G to the identity element of $\overline{G}$.

(f) What is meant by cycle of a permutation? Give an example.

(g) Show that in a group $(G, \bullet)$,

$$(a.b)^{-1} = b^{-1}.a^{-1}, \ a, b \in G.$$

(h) Define centre of a group G and give an example.

(i) Give an example of a group containing only three elements.

(j) Define group isomorphism and give an example.

3. Answer ***any four*** questions : 5×4=20

(a) Show that *any two* cycles of a permutation of a finite set are disjoint.

(b) If H and K are two normal subgroups of a group G such that $H \cap K = \{e\}$ (e being the identity element of G), then show that $hk = kh$ for all $h \in H, k \in K$.

(c) Let H be a subgroup of a group G. Show that there exists a one-one and onto map between the set of all left cosets of H in G and the set of all right cosets of H in G.

Contd.

(d) Let G be a group. If $a \in G$ is of finite order n and also $a^m = e$, then show that n/m.

(e) Let f be a homomorphism from a group G to a group G'. Show that $\ker f$ is a normal subgroup of G.

(f) If $\mathbb{R}^*$ is the group of non-zero real numbers under multiplication, then show that $(\mathbb{R}^*, \bullet)$ is not isomorphic to $(\mathbb{R}, +)$.

(g) Prove that a cyclic group is abelian.

(h) Consider the multiplicative group $G = \{1, -1, i, -i\}$. Define a self mapping ϕ on G which is a homomorphism and justify your answer.

4. Answer ***any four*** questions : 10×4=40

(a) Let G be a group. Show that

(i) the centre of G is a subgroup of G;

(ii) for each $a \in G$, the centralizer of a is a subgroup of G.

(b) Let G be a group in which

$(ab)^3 = a^3b^3$

$(ab)^5 = a^5b^5$ for all $a, b \in G$.

Prove that G is abelian.

(c) Prove that every subgroup of a cyclic group is cyclic. Also show that if $|\langle a \rangle| = n$, then the order of any subgroup of $\langle a \rangle$ is a divisor of n.

(d) If H and K are finite subgroups of a group G, then prove that

$$|HK| = \frac{|H|.|K|}{|H \cap K|}$$

(d) Let G be a group. If $a \in G$ is of finite order n and also $a^m = e$, then show that n/m.

(e) Let f be a homomorphism from a group G to a group G'. Show that ker f is a normal subgroup of G.

(f) If $\mathbb{R}^*$ is the group of non-zero real numbers under multiplication, then show that $(\mathbb{R}^*, \bullet)$ is not isomorphic to $(\mathbb{R}, +)$.

(g) Prove that a cyclic group is abelian.

(h) Consider the multiplicative group $G = \{1, -1, i, -i\}$. Define a self mapping ϕ on G which is a homomorphism and justify your answer.

4. Answer **any four** questions : 10×4=40

(a) Let G be a group. Show that

(i) the centre of G is a subgroup of G;

(ii) for each $a \in G$, the centralizer of a is a subgroup of G.

(b) Let G be a group in which

$(ab)^3 = a^3b^3$

$(ab)^5 = a^5b^5$ for all $a, b \in G$.

Prove that G is abelian.

(c) Prove that every subgroup of a cyclic group is cyclic. Also show that if $|\langle a \rangle| = n$, then the order of any subgroup of $\langle a \rangle$ is a division of n.

(d) If H and K are finite subgroups of a group G, then prove that

$$|HK| = \frac{|H|.|K|}{|H \cap K|}$$

(e) Prove that the order of a permutation of a finite set written in disjoint cycle form is the least common multiple of the lengths of the cycles.

(f) Let G be a finite abelian group and let p be a prime that divides the order of G. Prove that G has an element of order p.

(g) Let ϕ be an isomorphism from a group G onto a group $\overline{G}$. Prove that—

(i) for every integer n and for every $a \in G$, $\phi(a^n) = [\phi(a)]^n$;

(ii) $|a| = |\phi(a)|$ for all $a \in G$.

(h) State and prove the second isomorphism theorem for groups.

(i) Show that the order of a cyclic group is same as the order of its generator.

(j) Consider the multiplicative group $G = \{1, -1, i, -i\}$. Find all the subgroups of G and verify Lagrange's theorem for each subgroup.

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3 (Sem–3/CBCS) MAT HC 3

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–3036

(Analytical Geometry)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any ten*** : 1×10=10

   (i) Write down the formulae of transformation from one pair of rectangular axes to another with same origin.

   (ii) Find the equation to the locus of the point $P(t, 2t)$ if t is a parameter.

Contd.

Total number of printed pages–11

3 (Sem–3/CBCS) MAT HC 3

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–3036

(Analytical Geometry)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any ten*** : $1\times10=10$

(i) Write down the formulae of transformation from one pair of rectangular axes to another with same origin.

(ii) Find the equation to the locus of the point $P(t, 2t)$ if t is a parameter.

Contd.

(iii) For what value of a, the transformation $x' = -x + 2,\ y' = ax + 3$ is a translation ?

(iv) What is the locus represented by the equation $ax^2 - 5xy + 6y^2 = 0$?

(v) Write down the polar equation of the straight line $x = 0$.

(vi) Under what condition

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

may represents a pair of straight lines ?

(vii) What will be the equation of the line $ax + by + c = 0$ if the origin is transferred to the point (α, β)?

(viii) The parabola represented by the equation $y^2 = 4ax$ is not a closed curve. How can you justify it from the given equation ?

(ix) Write the relationship between the lengths of semi-major axis, semi-minor axis and the eccentricity for the standard equation of the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ b > a$.

(x) What conic does the following equation represent ?

$$x^2 + 2xy + y^2 - 2x - 1 = 0$$

(xi) What are the direction ratios of the normal to the plane given by equation $ax + by + cz + d = 0$?

(xii) Write down the direction cosines of z-axis.

(xiii) When does the equation $ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$ represent a sphere ?

(xiv) When is a plane said to be parallel to a line ?

(xv) Mention the condition under which the lines $\frac{x - \alpha_1}{l_1} = \frac{y - \beta_1}{m_1} = \frac{z - \gamma_1}{n_1}$ and

$\frac{x - \alpha_2}{l_2} = \frac{y - \beta_2}{m_2} = \frac{z - \gamma_2}{n_2}$ are coplanar.

(xvi) What are centre and radius of the sphere given by the equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$?

(xvii) Define the polar plane of a point (α, β, γ) with respect to the conicoid $ax^2 + by^2 + cz^2 = 1$.

(xviii) What are the coordinates of the vertex of the cone

$$ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0 \text{ ?}$$

2. Answer **any five** : 2×5=10

(a) Find the equation of the line $y = \sqrt{3}x$ when the axes are rotated through an angle $\frac{\pi}{3}$.

(b) If $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are the extremities of any focal chord of the parabola $y^2 = 4ax$, prove that $t_1t_2 = -1$.

(c) If the two pair of lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, prove that $pq + 1 = 0$.

(d) If e_1 and e_2 are the eccentricities of a hyperbola and its conjugate, show that

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1.$$

(e) Find the equation of the plane containing the lines $2x + 3y + 5z - 7 = 0$, $3x - 4y + z + 14 = 0$ and passing through the origin.

(f) Find the equation of the cone whose vertex is at the origin and whose guiding curve is given by $x = a,\ y^2 + z^2 = b^2$.

(g) Find the equation of the sphere through the circle $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ and the point $(1, 2, 3)$.

(h) Mention the conditions under which the general equation of the second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents *(i)* a parabola, *(ii)* an ellipse, *(iii)* a hyperbola, and *(iv)* a circle.

(i) Find the perpendicular distance of the point $(1, 4, -2)$ from the plane $2x - 3y + z = 5$.

(xvii) Define the polar plane of a point (α, β, γ) with respect to the conicoid $ax^2 + by^2 + cz^2 = 1$.

(xviii) What are the coordinates of the vertex of the cone

$$ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0 \text{ ?}$$

2. Answer **any five** : $2\times5=10$

(a) Find the equation of the line $y = \sqrt{3}x$ when the axes are rotated through an angle $\frac{\pi}{3}$.

(b) If $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are the extremities of any focal chord of the parabola $y^2 = 4ax$, prove that $t_1t_2 = -1$.

(c) If the two pair of lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, prove that $pq + 1 = 0$.

(d) If e_1 and e_2 are the eccentricities of a hyperbola and its conjugate, show that

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1.$$

(e) Find the equation of the plane containing the lines $2x + 3y + 5z - 7 = 0$, $3x - 4y + z + 14 = 0$ and passing through the origin.

(f) Find the equation of the cone whose vertex is at the origin and whose guiding curve is given by

$$x = a,\ y^2 + z^2 = b^2.$$

(g) Find the equation of the sphere through the circle

$$x^2 + y^2 + z^2 = 9,\ 2x + 3y + 4z = 5$$

and the point (1, 2, 3).

(h) Mention the conditions under which the general equation of the second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents (i) a parabola, (ii) an ellipse, (iii) a hyperbola, and (iv) a circle.

(i) Find the perpendicular distance of the point (1, 4, −2) from the plane $2x - 3y + z = 5$.

(j) The axis of a right circular cylinder is

$$\frac{x-1}{2}=\frac{y-2}{-1}=\frac{z-3}{2}$$ and its radius is 5.

Find its equation.

3. Answer ***any four*** : 5×4=20

(a) Prove that the transformation of rectangular axes which converts

$$\frac{X^2}{P}+\frac{Y^2}{Q}$$ into $ax^2+2hxy+by^2$ will

convert $\frac{X^2}{P-\lambda}+\frac{Y^2}{Q-\lambda}$ into

$$\frac{ax^2+2hxy+by^2-\lambda\left(ab-h^2\right)\left(x^2+y^2\right)}{1-(a+b)\lambda+\left(ab-h^2\right)\lambda^2}.$$

(b) Prove that the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents a pair of parallel straight lines if $\frac{a}{h}=\frac{h}{b}=\frac{g}{f}$.

(c) Show that the line $lx+my=n$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, if $a^2l^2+b^2m^2=n^2$.

(d) Prove that the product of the perpendiculars from any point on a hyperbola to the asymptotes is constant.

(e) A plane passes through a fixed point (p, q, r), and cut the axes in A, B, C. Show that the locus of the centre of the sphere $OABC$ is $\frac{p}{x}+\frac{q}{y}+\frac{r}{z}=2$.

(f) Find the centre and radius of the circle

$$x^2+y^2+z^2-8x+4y+8z-45=0,$$

$$x-2y+2z=3.$$

(g) The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the coordinate axes in A, B, C. Prove that the equation of the cone generated by the lines drawn from O is

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

(h) Find the equation of the circular cylinder whose guiding circle is

$x^2 + y^2 + z^2 = 9, x - y + z = 3.$

4. Answer **any four** : 10×4=40

(a) If the equation

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

represents a pair of straight lines, prove that the equation to the third pair of straight lines passing through the points, where these meet the axes is

$$ax^2 - 2hxy + by^2 + 2gx + 2fy + c + \frac{4fg}{c}xy = 0.$$

(b) Obtain the equation of the chord of the conic $\frac{l}{r} = 1 + e\cos\theta$, joining the two points on the conic, whose vectorial angles are $(\alpha + \beta)$ and $(\alpha - \beta)$.

(c) If PSP' and QSQ' are two perpendicular focal chords of a conic, prove that

$$\frac{1}{PS.SP'} + \frac{1}{QS.SQ'} = \text{a constant.}$$

(d) Find the equation of a polar of a given point $P(x_1, y_1)$ with respect to the conic

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$

Hence show that if the polar of a point P with respect to the conic passes through a point Q, then the polar of Q also passes through P.

(e) (i) Show that the locus of the points of intersection of perpendicular tangents to a parabola is its directrix.

(ii) Find the asymptotes of the hyperbola $xy + ax + by = 0$.

(f) What do you mean by skew lines? How do you define the shortest distance between two such lines? Find the length and the equations of the line of shortest distance between the lines

$3x - 9y + 5z = 0, x + y - z = 0$

$6x + 8y + 3z - 13 = 0, x + 2y + z - 3 = 0.$

(h) Find the equation of the circular cylinder whose guiding circle is

$$x^2 + y^2 + z^2 = 9,\ x - y + z = 3.$$

4. Answer **any four** : 10×4=40

(a) If the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a pair of straight lines, prove that the equation to the third pair of straight lines passing through the points, where these meet the axes is

$$ax^2 - 2hxy + by^2 + 2gx + 2fy + c + \frac{4fg}{c}xy = 0.$$

(b) Obtain the equation of the chord of the conic $\frac{l}{r} = 1 + e\cos\theta$, joining the two points on the conic, whose vectorial angles are $(\alpha + \beta)$ and $(\alpha - \beta)$.

(c) If PSP' and QSQ' are two perpendicular focal chords of a conic, prove that

$$\frac{1}{PS.SP'} + \frac{1}{QS.SQ'} = \text{a constant.}$$

(d) Find the equation of a polar of a given point $P(x_1, y_1)$ with respect to the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Hence show that if the polar of a point P with respect to the conic passes through a point Q, then the polar of Q also passes through P.

(e) (i) Show that the locus of the points of intersection of perpendicular tangents to a parabola is its directrix.

(ii) Find the asymptotes of the hyperbola $xy + ax + by = 0$.

(f) What do you mean by skew lines? How do you define the shortest distance between two such lines? Find the length and the equations of the line of shortest distance between the lines

$$3x - 9y + 5z = 0,\ x + y - z = 0$$
$$6x + 8y + 3z - 13 = 0,\ x + 2y + z - 3 = 0.$$

(g) Prove that the radius of the circle in which the plane

$$\frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2}=\lambda \text{ cuts the}$$

ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is

$$b\sqrt{1-\frac{\lambda^2}{a^2-c^2}}.$$

(h) Prove that the lines through (α, β, γ) at right angles to their polars with respect to $\frac{x^2}{a+b}+\frac{y^2}{2a}+\frac{z^2}{2b}=1$ generate the cone

$$(y-\beta)(\alpha z-\gamma x)+(z-\gamma)(\alpha y-\beta x)=0.$$

What is the peculiarity of the case when $a=b$?

(i) Show that the equation of the cylinder whose generators are parallel to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ and guiding curve is

$x^2+2y^2=1,\ z=3$ is

$$3(x^2+2y^2+z^2)+8yz-2zx+6x-24y-18z+24=0.$$

(j) What do you mean by a director sphere? Find the equation of the director sphere of the conicoid $ax^2+by^2+cz^2=1$. Hence or otherwise prove that the director sphere of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is

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