Total number of printed pages–11

3 (Sem–2/CBCS) MAT HC 1

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–2016

(Real Analysis)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any ten*** questions : 1×10=10

(a) Find the infimum of the set

$$\left\{1-\frac{(-1)^n}{n} : n \in N\right\}$$

(b) If A and B are two bounded subsets of $\mathbb{R}$, then which one of the following is true?

(i) $sup(A \cup B) = sup\{supA, supB\}$

(ii) $sup(A \cup B) = sup\, A + sup\, B$

Contd.

(iii) $sup(A \cup B) = sup\ A \,.\, sup\ B$

(iv) $sup(A \cup B) = sup\ A \cup sup\ B$

(c) There does not exist a rational number x such that $x^2 = 2$. *(Write True or False)*

(d) The set Q of rational numbers is uncountable. *(Write True or False)*

(e) If $I_n = \left(0, \frac{1}{n}\right)$ for $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} I_n = ?$

(f) The convergence of $\{|x_n|\}$ imply the convergence of $\{x_n\}$.

(Write True or False)

(g) What are the limit points of the sequence $\{x_n\}$, where $x_n = 2 + (-1)^n,\ n \in \mathbb{N}$?

(h) If $\{x_n\}$ is an unbounded sequence, then there exists a properly divergent subsequence. *(Write True or False)*

(i) A convergent sequence of real numbers is a Cauchy sequence.

(Write True or False)

(j) If $0 < a < 1$ then $\lim_{n \to \infty} a^n = ?$

(k) The positive term series $\sum \frac{1}{n^p}$ is convergent if and only if

(i) $p > 0$

(ii) $p > 1$

(iii) $0 < p < 1$

(iv) $p \leq 1$

(Write correct one)

(l) Define conditionally convergent of a series.

(m) If $\{x_n\}$ is a convergent monotone sequence and the series $\sum_{n=1}^{\infty} y_n$ is convergent, then the series $\sum_{n=1}^{\infty} x_n y_n$ is also convergent.

*(Write True **or** False)*

(n) Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^m\left(1+\frac{1}{n^p}\right)}$

where m and p are real numbers under which of the following conditions does the above series convergent ?

(i) $m > 1$

(ii) $0 < m < 1$ and $p > 1$

(iii) $0 \le m \le 1$ and $0 \le p \le 1$

(iv) $m = 1$ and $p > 1$

(o) Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers defined by $x_1 = 1$, $y_1 = \frac{1}{2}$,

$x_{n+1} = \frac{x_n + y_n}{2}$ and $y_{n+1} = \sqrt{x_n y_n}$ $\forall n \in \mathbb{N}$

then which one of the following is true ?

(i) $\{x_n\}$ is convergent, but $\{y_n\}$ is not convergent

(ii) $\{x_n\}$ is not convergent, but $\{y_n\}$ is convergent

(iii) Both $\{x_n\}$ and $\{y_n\}$ are convergent and $\lim_{n\to\infty} x_n > \lim_{n\to\infty} y_n$

(iv) Both $\{x_n\}$ and $\{y_n\}$ are convergent and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$

2. Answer ***any five*** parts : 2×5=10

(a) If a and b are real numbers and if $a < b$, then show that $a < \frac{1}{2}(a+b) < b$.

(b) Show that the sequence $\left\{\frac{2n-7}{3n+2}\right\}$ is bounded.

(c) If $\{x_n\}$ converges in $\mathbb{R}$, then show that $\lim_{n\to\infty} x_n = 0$

(d) Show that the series 1+2+3+...., is not convergent.

(e) Test the convergence of the series :

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots.$$

(f) State Cauchy's integral test of convergence.

(g) State the completeness property of $\mathbb{R}$ and find the $sup\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

(h) Does the Nested Interval theorem hold for open intervals ? Justify with a counter example.

3. Answer ***any four*** parts : 5×4=20

(a) If x and y are real numbers with $x < y$, then prove that there exists a rational number r such that $x < r < y$.

(b) Show that a convergent sequence of real numbers is bounded.

(c) Prove that $\lim\limits_{n \to \infty}\left(n^{\frac{1}{n}}\right) = 1$.

(d) $\{x_n\}$ be a sequence of real numbers that converges to x and suppose that $x_n \geq 0$. Show that the sequence $\{\sqrt{x_n}\}$ of positive square roots converges and $\lim\limits_{n \to \infty} \sqrt{x_n} = \sqrt{x}$.

(e) Show that every absolutely convergent series is convergent. Is the converse true ? Justify. 4+1=5

(f) Using comparison test, show that the series $\sum\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$ is convergent.

(g) State Cauchy's root test. Using it, test the convergence of the series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots\ldots$$

1+4=5

(h) Show that the sequence defined by the recursion formula

$$u_{n+1}=\sqrt{3u_n},\ u_1=1$$

is monotonically increasing and bounded. Is the sequence convergent ? 2+2+1=5

4. Answer ***any four*** parts : 10×4=40

(a) Show that the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is convergent and $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ which lies between 2 and 3.

(b) *(i)* Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences of real numbers such that $x_n \le y_n \le z_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n$.

Show that $\{y_n\}$ is convergent and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n$ 5

(ii) What is an alternating series ? State Leibnitz's test for alternating series. Prove that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots\infty$ is a conditionally convergent series. 1+1+3=5

(c) Test the convergence of the series

$1+a+a^2+\ldots\ldots+a^n+\ldots\ldots$

(d) (i) Using Cauchy's condensation test, discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ 5

(ii) Define Cauchy sequence of real numbers. Show that the sequence $\left\{\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+.....+\frac{1}{n!}\right\}$ is a Cauchy sequence. 1+4=5

(e) (i) Show that a convergent sequence of real numbers is a Cauchy sequence. 5

(ii) Using Cauchy's general principle of convergence, show that the sequence $\left\{1+\frac{1}{2}+.....+\frac{1}{n}\right\}$ is not convergent. 5

(f) (i) Prove that every monotonically increasing sequence which is bounded above converges to its least upper bound. 5

(ii) Show that the limit if exists of a convergent sequence is unique. 5

(g) State and prove p-series.

(h) (i) Test the convergence of the series

$$x+\frac{3}{5}x^2+\frac{8}{10}x^3+.....+\frac{n^2-1}{n^2+1}x^n+.....(x>0)$$

5

(ii) If $\{x_n\}$ is a bounded increasing sequence then show that

$\lim_{n\to\infty} x_n = sup\{x_n\}$ 5

(i) (i) Show that a bounded sequence of real numbers has a convergent subsequence. 5

(ii) State and prove Nested Interval theorem. 5

(j) (i) Show that Cauchy sequence of real numbers is bounded. 5

(ii) Test the convergence of the series

$$x^2 + \frac{2^2}{3.4}x^4 + \frac{2^2.4^2}{3.4.5.6}x^6 + \frac{2^2.4^2.6^2}{3.4.5.6.7.8}x^8 + \ldots\ldots (x > 0)$$

5

Total number of printed pages–8

3 (Sem–2/CBCS) MAT HC 2

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–2026

(Differential Equations)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : ***(any seven)*** 1×7=7

   (a) Mention *one* principal goal of study of differential equations.

   (b) Write down the order of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^3 + 3\frac{dy}{dx} + y = e^x$$

Contd.

(c) What is meant by implicit solution of a differential equation ?

(d) Find the integrating factor of the linear differential equation

$$\frac{dy}{dx}+\left(\frac{2x+1}{x}\right)y=e^{-2x}$$

(e) State the processes in which compartmental model technique is used to formulate the mathematical model.

(f) Draw the input-output compartmental diagram for CO_2 in earth's atmosphere mentioning the compartment in the model.

(g) Define Wronskian of two functions f and g. What is its value if f and g are linearly dependent ?

(h) The roots of the characteristic equation of a certain differential equation are $0,0,0,3,-4,2\pm 3i$. Write a general solution of the homogeneous differential equation.

(i) Write down the appropriate form of a particular solution of the differential equation

$$y''+3y'+4y=3x+2$$

(j) Find the general solution of the differential equation $y''-9y=0$.

2. Answer the following questions : ***(any four)*** 2×4=8

(a) Determine all values of the constant r for which $y = e^{rx}$ is a solution of $3y'' + 3y' - 4y = 0$.

(b) A function $y = g(x)$ is described by the property that the line tangent to the graph of g at the point (x, y) intersects the x-axis at the point $\left(x/2, 0\right)$. Write a differential equation of the form $\frac{dy}{dx} = f(x, y)$ having the function g as its solution.

(c) Find a particular solution of $y'' - 4y = 2e^{3x}$.

(d) State the assumptions made in developing a model of exponential decay and radioactivity.

(e) Reduce the Bernoulli equation

$$\frac{dy}{dx} + \frac{y}{2x} = \frac{x}{y^3}$$

to linear equation by appropriate transformation.

(f) Verify that $y_1 = 1$ and $y_2 = e^{3x}$ are solutions of the differential equation

$$y'' - 3y' = 0.$$

Also write the general solution of the equation.

(g) Obtain the transformations that can be used to reduce the equation

$$(x - 2y + 1)dx + (4x - 3y - 6)dy = 0$$

to a homogeneous equation.

(h) Determine whether the functions $f(x) = sin^2 x$ and $g(x) = 1 - cos 2x$ are linearly independent or linearly dependent on the real line.

3. Answer the following questions : ***(any three)*** 5×3=15

(a) Solve the equation

$$(x^2 - 3y^2)dx + 2xy\,dy = 0$$

(b) Solve the initial value problem

$\frac{dy}{dx} + y = f(x)$ where

$$f(x) = \begin{cases} 2, & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases} \quad y(0) = 0$$

(c) Find a general solution of the differential equation $\frac{dy}{dx} = y^2$.

Find a singular solution of the equation mentioning why is it a singular solution ? 3+2=5

(d) State *one* situation where limited growth of population with harvesting model can be useful.
Formulate the differential equation for such model. 2+3=5

(e) Verify that the functions $y_1(x) = e^x$ and $y_2(x) = xe^x$ are solutions of the differential equation $y'' - 2y' + y = 0$. Also find a solution satisfying the initial conditions $y(0) = 3, y'(0) = 1$. 3+2=5

(f) The rate at which earth's atmospheric pressure p changes with altitude h above sea level is proportional to p. Suppose that the pressure at sea level is 1,013 milibars and that the pressure at an altitude of 20 *km* is 50 milibars. Use an exponential decay model.

$\frac{dp}{dh} = -kp$ to describe the system, and then solving the equation find an expression for p in terms of h. Determine k and the constant of integration from the initial conditions. What is the atmospheric pressure at an altitude of 50 *km* ? 2+2+1=5

Contd.

(g) Use the method of variation of parameters to find a particular solution of the differential equation

$$y'' + 3y' + 2y = 4e^x$$

(h) Solve the initial value problem

$y^{(3)} + 3y'' - 10y' = 0$, given

$y(0) = 7,\ y'(0) = 0,\ y''(0) = 70$

4. Answer the following questions : ***(any three)*** 10×3=30

(a) Consider the differential equation

$(y^2 + 2xy)dx - x^2 dy = 0$

(i) Show that this equation is not exact.

(ii) Multiply the given equation through by y^n, where n is an integer and then determine n so that y^n is an integrating factor of the given equation.

(iii) Multiply the given equation through by the integrating factor found in (ii) and solve the resulting exact equation.

(iv) Show that $y = 0$ is a solution of the original non exact equation but is not a solution of the essentially equivalent exact equation.

(v) Name the solution $y = 0$ of the given equation. 2+2+4+1+1=10

(b) Suppose the velocity v of a motor boat coasting in water satisfies the differential equation $\frac{dv}{dt} = kv^2$. The initial speed of the motorboat is $v(0) = 10 m/sec$ and v is decreasing at the rate of 1 m/sec^2 when $v = 5$ m/sec. How long does it take for the velocity of the boat to decrease to 1 m/sec ? To $1/10$ m/sec ? When does the boat come to a stop ? 6+2+2=10

(c) Mentioning the assumptions used, formulate the differential equation for exponential growth or decay in population model.
Obtain the solution of the equation. Find an expression for the time for the population to double in size.
5+3+2=10

(d) Solve the initial value problem

$$y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x$$

Given $y(0) = 1$, $y'(0) = 2$

(e) Explain the need of density dependent growth model of population over exponential growth model. Formulate the differential equation for density dependent growth model of population.
5+5=10

(f) Find the general solution of

(i) $y^{(4)} = 16y$

(ii) $y^{(3)} + 3y'' + 4y' - 8y = 0$ 4+6=10

(g) (i) Use method of underlined co-efficient to solve $y'' - y' - 2y = 3x + 4$

(ii) Solve the Euler equation

$$x^2y'' + xy' + 9y = 0$$

by appropriate transformations. 5+5=10

(h) The population of a certain city satisfies the logistic law

$$\frac{dx}{dt} = \frac{1}{100}x - \frac{1}{(70)^8}x^2$$

Where time t is measured in years. Given that the population of this city is 1,00,000 in 1980, determine the population as a function of time for $t > 1980$. Hence answer the following :

(i) What will be the population in 2030 ?

(ii) By which year does the 1980 population double ? 6+2+2=10