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3 (Sem–6/CBCS) MAT HC 1

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–6016

(Complex Analysis)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any seven*** questions from the following : $1 \times 7 = 7$

   (a) If c is any nth root of unity other than unity itself, then value of $1 + c + c^2 + \cdots + c^{n-1}$ is

   (i) $2n\pi$

   (ii) 0

   (iii) –1

   (iv) None of the above

   (Choose the correct answer)

Contd.

(b) The square roots of $2i$ is

(i) $\pm(1+i)$

(ii) $\pm(1-i)$

(iii) $\pm\frac{1}{\sqrt{2}}\left(1-i\sqrt{2}\right)$

(iv) None of the above

(Choose the correct answer)

(c) A composition of continuous function is

(i) discontinuous

(ii) itself continuous

(iii) pointwise continuous

(iv) None of the above

(Choose the correct answer)

(d) The value of $Log(-ei)$ is

(i) $\frac{\pi}{2}-i$

(ii) i

(iii) $1-\frac{\pi}{2}i$

(iv) None of the above

(Choose the correct answer)

(e) The power expression of $\cos z$ is

(i) $\dfrac{e^{z}+e^{-z}}{2}$

(ii) $\dfrac{e^{iz}+e^{-iz}}{2}$

(iii) $\dfrac{e^{iz}+e^{-iz}}{2i}$

(iv) None of the above

(Choose the correct answer)

(f) The Cauchy-Riemann equation for analytic function $f(z)=u+iv$ is

(i) $u_x = v_y\ ,\ u_y = -v_x$

(ii) $u_x = -v_y\ ,\ u_y = v_x$

(iii) $u_{xx} + v_{yy} = 0$

(iv) None of the above

(Choose the correct answer)

(g) If $w(t)=u(t)+iv(t)$, then $\dfrac{d}{dt}[w(t)]^2$ is equal to

(i) $2[u(t)+iv(t)]$

(ii) $2w'(t)$

(iii) $2w(t)w'(t)$

(iv) None of the above

(Choose the correct answer)

(h) What is Laplace's equation?

(i) What is extended complex plane?

(j) What is Jordan arc?

2. Answer ***any four*** questions from the following: 2×4=8

(a) Write principal value of $arg\left(\frac{i}{-1-i}\right)$.

(b) If $f(z) = x^2 + y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$, then write $f(z)$ in terms of z.

(c) Use definition to show that $\lim_{z \to z_0} \bar{z} = \bar{z}_0$.

(d) Find the singular point of

$$f(z) = \frac{z^2 + 3}{(z+1)(z^2+5)}.$$

(e) If $f'(z) = 0$ everywhere in a domain D, then prove that $f(z)$ must be constant throughout D.

(f) Evaluate $f'(z)$ from definition, where $f(z)=\frac{1}{z}$.

(g) If $f(z)=\frac{z}{\bar{z}}$, find $\lim_{z\to 0} f(z)$, if it exists.

(h) Write the function $f(z)=z+\frac{1}{z}\ (z\neq 0)$ in the form $f(z)=u(r,\theta)+iv(r,\theta)$.

3. Answer **any three** questions from the following : 5×3=15

(a) If z_1 and z_2 are complex numbers, then show that
$\sin(z_1+z_2)=\sin z_1 \cos z_2+\cos z_1 \sin z_2$.

(b) Show that exp. $(2\pm 3\pi i)=-e^2$.

(c) Sketch the set $|z-2+i|\leq 1$ and determine its domain.

(d) Let C be the arc of the circle $|z|=2$ from $z=2$ to $z=2i$, that lies in the 1st quadrant, then show that

$$\left|\int_C \frac{z-2}{z^2+1}\,dz\right|\leq \frac{4\pi}{15}$$

(e) Evaluate $\int_C \frac{dz}{z}$, where C is the top half of the circle $|z|=1$ from $z=1$ to $z=-1$.

(f) If $f(z)=e^z$, then show that it is an analytic function.

(g) If $f(z)=\frac{z+2}{z}$ and C is the semi circle $z=2e^{i\theta}$, $(0 \le \theta \le \pi)$, then evaluate $\int_C f(z)\,dz$.

(h) Find all values of z such that $e^z=-2$.

4. Answer ***any three*** questions from the following : 10×3=30

(a) State and prove Cauchy-Riemann equations of an analytic function in polar form.

(b) Suppose that

$$f(z)=u(x, y)+iv(x, y),\ (z=x+iy)$$

and $z_0=x_0+iy_0$, $w_0=u_0+iv_0$, then prove that if $\lim\limits_{(x,y)\to(x_0,y_0)} u(x, y)=u_0$ and $\lim\limits_{(x,y)\to(x_0,y_0)} v(x, y)=v_0$ then $\lim\limits_{z\to z_0} f(z)=w_0$ and conversely.

(c) If the function $f(z)=u(x,y)+iv(x,y)$ is defined by means of the equation

$$f(z)=\begin{cases}\dfrac{\overline{z}^{e}}{z}, & \text{when } z\neq 0\\ 0, & \text{when } z=0,\end{cases}$$

then prove that its real and imaginary parts satisfies Cauchy-Riemann equations at $z=0$. Also show that $f'(0)$ fails to exist.

(d) If the function $f(z)=u(x,y)+iv(x,y)$ and its conjugate $\bar{f}(z)=u(x,y)-iv(x,y)$ are both analytic in a domain *D*, then show that $f(z)$ must be constant throughout *D*.

(e) If *f* be analytic everywhere inside and on a simply closed contour *C*, taken in the positive sense and z_0 is any point interior to *C*, then prove that

$$f(z_0)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz.$$

(f) State and prove Liouville's theorem.

(g) Suppose that a function f is analytic throughout a disc $|z - z_0| < R_0$ centred at z_0 and with radius R_0. Then prove that $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad (|z - z_0| < R_0)$$

where $a_n = \dfrac{f^n(z_0)}{\underline{|n}}, \quad (n = 0, 1, 2, \ldots\ldots)$

(h) State and prove Laurent's theorem.

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3 (Sem–6/CBCS) MAT HC 2

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–6026

(Partial Differential Equations)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any seven*** : 1×7=7

(i) The equation of the form

$P_p + Q_q = \mathbb{R}$ is known as

(a) Charpit's equation

(b) Lagrange's equation

(c) Bernoulli's equation

(d) Clairaut's equation

(Choose the correct answer)

Contd.

(ii) How many minimum no. of independent variables does a partial differential equation require?

(iii) Find the degree and order of the equation

$$\frac{\partial^3 z}{\partial x^3}+\left(\frac{\partial^3 z}{\partial x \partial y^2}\right)^2+\frac{\partial z}{\partial y}=sin(x+2y)$$

(iv) Which method can be used for finding the complete solution of a non-linear partial differential equation of first order

(a) Jacobi method

(b) Charpit's method

(c) Both *(a)* and *(b)*

(d) None of the above

(Choose the correct answer)

(v) State *True* **Or** *False* :

The equation

$$u_{xx}+u_{yy}+u_{zz}=0$$

is an Hyperbolic equation.

(vi) Fill in the blanks :

$$\left(\frac{\partial z}{\partial x}\right)^2+2\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}+z=0$$

is a ______ ______ order partial differential equation.

(vii) The characteristic equation of $yu_x + xu_y = u$ is

(a) $\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}$

(b) $\frac{dx}{y} = \frac{dy}{x} = \frac{du}{u}$

(c) $\frac{dx}{u} = \frac{dy}{x} = \frac{du}{y}$

(d) None of the above

(Choose the correct answer)

(viii) State *True* **Or** *False*

$xu_x + yu_y = u^2 + x^2$ is a semi-linear partial differential equation.

(ix) Fill in the blanks :

A solution $z = z(x, y)$ when interpreted as a surface in 3-dimensional space is called ______ ______.

(x) The partial differential equation is elliptical if

(a) $B^2 - 4AC > 0$

(b) $B^2 - 4AC \geq 0$

(c) $B^2 - 4AC \leq 0$

(d) $B^2 - 4AC < 0$

(Choose the correct answer)

2. Answer ***any four*** : 2×4=8

(i) Define quasi-linear partial differential equation and give *one* example.

(ii) Show that a family of spheres $(x-a)^2+(y-b)^2=r^2$ satisfies the partial differential equation
$$z^2\left(p^2+q^2+1\right)=r^2$$

(iii) Eliminate the constants a and b from $z=(x+a)(y+b)$.

(iv) Determine whether the given equation is hyperbolic, parabolic or elliptic
$$u_{xx}-2u_{yy}=0.$$

(v) Solve the differential equation $p+q=1$.

(vi) Explain the essential features of the "Method of separation of variables".

(vii) Mention when Charpit's method is used. Name a disadvantage of Charpit's method.

(viii) What is the classification of the equation
$$u_{xx}-4u_{xy}+4u_{yy}=e^{y}$$

3. Solve ***any three*** : 5×3=15

(i) Form a partial differential equation by eliminating arbitrary functions f and F from $y = f(x - at) + F(x + at)$.

(ii) Solve

$$y^2 p - xyq = x(z - 2y)$$

(iii) Find the integral surface of the linear partial differential equation

$$x\left(y^2 + z\right)p - y\left(x^2 + z\right)q = \left(x^2 - y^2\right)z$$

which contains the straight line $x + y = 0,\ z = 1$.

(iv) Find the solution of the equation $z = pq$ which passes through the parabola $x = 0,\ y^2 = z$.

(v) Find a complete integral of the equation $x^2 p^2 + y^2 q^2 = 1$.

(vi) Reduce the equation $yu_x + u_y = x$ to canonical form and obtain the general solution.

(vii) Apply the method of separation of variables $u(x, y) = f(x)\, g(y)$ to solve the equation $u_x + u = u_y$,

$u(x, 0) = 4e^{-3x}$.

(viii) Determine the general solution of

$4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$.

4. Answer ***any three*** : 10×3=30

(i) Solve $\left(p^2 + q^2\right)y - qz = 0$ by Jacobi method.

(ii) Solve $z^2 = pqxy$ by Charpit's method.

(iii) Find the general solution of the differential equation

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x + y)z$$

(iv) Solve

$$(mz - ny)p + (nx - lz)q = ly - mx$$

(v) Use $v = \ln u$ and $v = f(x) + g(y)$ to solve the equation

$$x^2 u_x^2 + y^2 u_y^2 = u^2.$$

(vi) Find the solution of the equation

$$z = \frac{1}{2}\left(p^2 + q^2\right) + (p - x)(q - y)$$

which passes through the x axis.

(vii) Find the canonical form of the equation

$$y^2 u_{xx} - x^2 u_{yy} = 0.$$

(viii) Classify the second order linear partial differential equation with example.
